Department of Economics- FEA/USP

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Graziella Magalhães
David Turchick

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Graziella Magalhães (graziella.magalhaes@ufv.br)

David Turchick (dturchick@usp.br)

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We study the impacts of different educational regimes on growth and income inequality using a two-stage human capital model with heterogeneous agents that takes the hierarchical nature of education into account. The differentiation of educational stages sheds new light on the impacts of human capital accumulation on growth and inequality. Both at the basic (elementary and secondary) and the advanced (higher) educational stages, the school system may be either public or private. Our analysis shows that the educational regime with the highest growth rate has private basic education. The completely public (private) regime is the one in which inequality vanishes the fastest (slowest), albeit leading to the lowest (highest) growth. If the government is to fund only one educational stage, such a decision will hinge on elasticities of the human capital production function and on the interest rate.

Keywords: human capital, growth, inequality, educational regime, hierarchical education.

JEL Codes: O40, O15, I24.

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Graziella Magalhães* David Turchick†

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*Universidade de São Paulo / Universidade Federal de Viçosa. Email address: graziella.magalhaes@ufv.br.
†Universidade de São Paulo. Email address: dturchick@usp.br.

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1 Introduction

This paper studies the long-term impacts of different educational regimes on economic growth and income inequality using a two-stage human capital accumulation model with heterogeneous agents. Our framework accounts for the hierarchical nature of education, in the sense that, the more basic math a student does in high school, the more he/she will be able to take from a Calculus or Linear Algebra course in college.[1] In each stage of the learning process, education expenditures may be either public or private. Our goal will be to understand which form of educational regime results in higher economic growth and the fastest reduction of inequality.

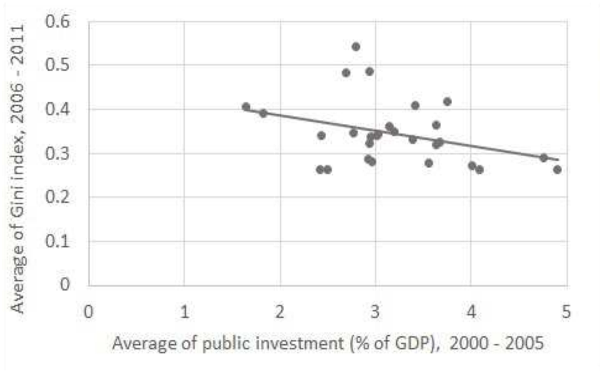
Different stages for human capital accumulation are considered: basic (elementary and secondary) education and advanced (higher) education. The importance of considering multiple educational stages derives from the inability of models with a single stage to capture the hierarchical nature of education, treating investment in schooling throughout the whole learning phase as perfect substitutes. However, the empirical evidence suggests that skill formation is a complex process and that not only are there critical periods for acquiring specific bits of knowledge, but also there is dynamic complementarity between investments in education at different stages (see Cunha and Heckman, 2007).

The average human capital of the economy (equal to teachers' human capital in our model) will impose an externality at each stage of the human capital accumulation process. This (positive) externality could justify public intervention in the educational sector. Therefore, we consider that, at each stage, the school system may be of either one of two types: public or private. This gives rise to four different educational regimes: completely private, completely public, and two mixed regimes (the private basic and public advanced regime, and the public basic and private advanced regime).

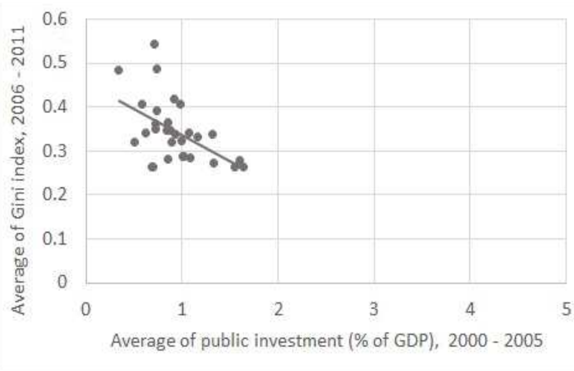
OECD data indicate that the correlation between investment in education and income inequality varies with educational stage and source of educational spending. In Figure 1, we plot the

[1] Cf. Su (2004), where it is assumed that, if a certain threshold was not met in learning basic math, then it is just impossible to learn more advanced math.

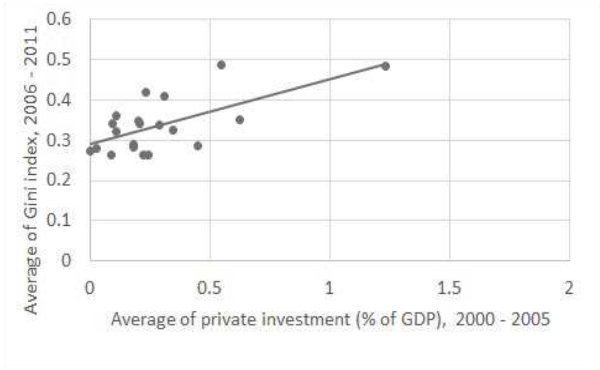
average public and private investment in education as percentage of GDP both for the basic and the advanced educational stages between 2000 and 2005 against the average of the Gini index between 2006 and 2011. As we can see, there is a negative relationship between public investment in education and the Gini index, and this relationship seems more pronounced for higher education (in the sense that the gray lines in the figure have different slopes). However, if one considers private investments in education and income inequality, there is a positive relationship, this time more pronounced at the elementary/secondary stage.



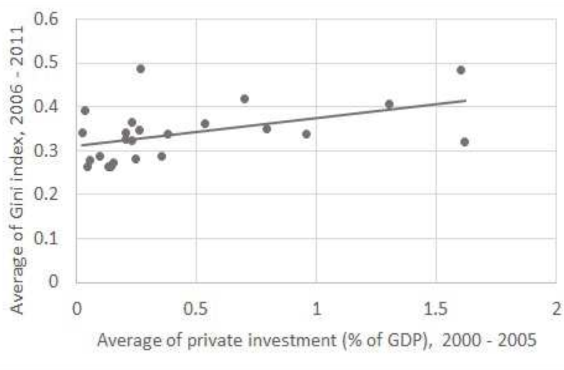
(a) Public spending on elementary/secondary education



(b) Public spending on higher education



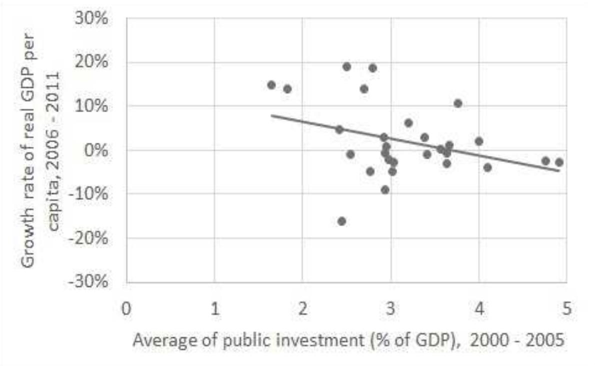
(c) Private spending on elementary/secondary education



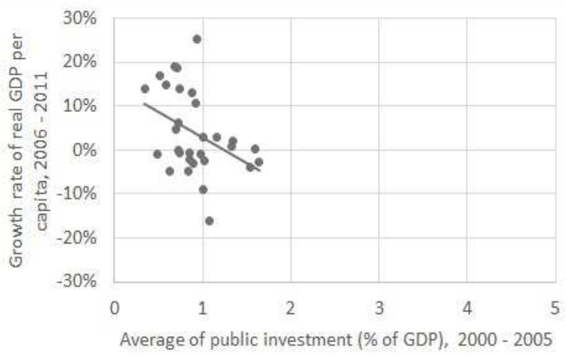
(d) Private spending on higher education

Figure 1: Public and private spending on education and income inequality

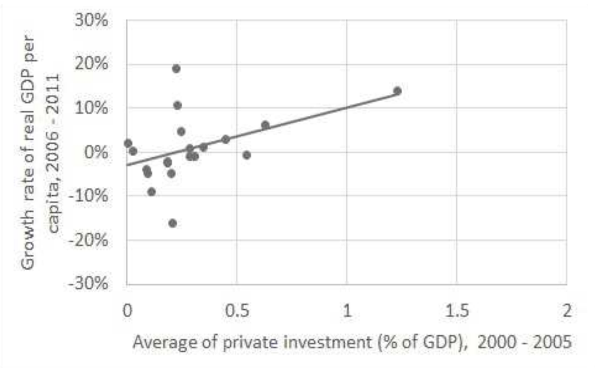
Figure 2 repeats this analysis, but with inequality replaced by GDP growth (more precisely, the growth rate of real GDP per capita in US dollars from 2006 to 2011). Comparing types of educational spending, the data show a negative correlation between public investments, in both educational stages, and economic growth. However, the correlation is positive when considering private investments. Again, it is worth noting that, although the direction of the relationship between educational investments and growth does not vary across stages, it steepness does. This simple exercise adds to the argument of considering educational investments separated by educational stage and their effects on inequality and growth.



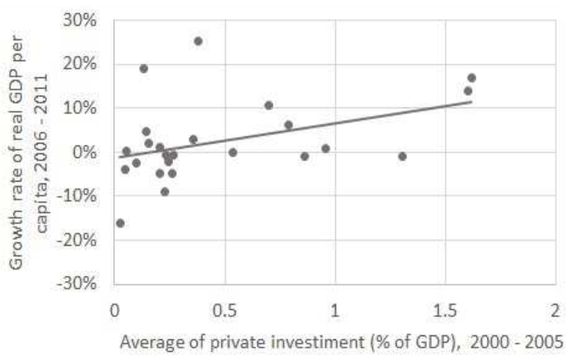
(a) Public spending on elementary/secondary education



(b) Public spending on higher education



(c) Private spending on elementary/secondary education



(d) Private spending on higher education

Figure 2: Public and private spending on education and economic growth

Throughout the long-standing and diverse literature on the link between inequality and growth (e.g., Kuznets, 1955; Caselli and Ventura, 2000; García-Penalosa and Turnovsky, 2006) and the specific strand considering human capital as the engine of growth (e.g., Tamura, 1991; de la Croix and Doepke, 2003, 2004; Galor, 2011), few recognize the importance of the hierarchical nature of education for the human capital accumulation process. Most of the recent literature considering a hierarchical education system focuses on the effects of investments across educational stages on the level of GDP and economic growth (Driskill and Horowitz, 2002; Arcalean and Schiopu, 2010;

Abington and Blankenau, 2013; Sarid, 2017), and some others on inequality (Su, 2004; Restuccia and Urrutia, 2004). Here we will compare different educational regimes with respect to their effects on both economic growth and inequality, in a framework that accounts for the hierarchical nature of education.

Additionally, we allow for decisions on fertility and on investments in education to be interdependent (as in Becker, Murphy and Tamura, 1990; de la Croix and Doepke, 2003, 2004). Due to a quality-quantity tradeoff of fertility, parents are expected to have more children under the completely public regime. This will be the case in our model, since parents have no say in the time investment in education, decided solely by the government. The fertility differential among regimes will rely upon the elasticities of human capital production regarding the investments in basic and advanced education. In a regime where some educational stage is private, the elasticity of human capital with respect to school-time investment at that stage negatively affects fertility.

These elasticities will also have a direct impact on the speed of reduction of inequality. As we shall see, the completely public regime is the one under which inequality vanishes the fastest, since investment in all students at the same educational stage is the same. In contrast, reduction of inequality will be the slowest possible, or even not take place at all, under the completely private regime, since wealthier parents will want their offspring to spend more time at school.

As for the mixed educational regimes, the one that presents the fastest reduction of inequality is the regime in which the government funds that educational stage where school-time investments are the most productive in the formation of human capital. Our calibration will indicate, in line with the literature on the economics of education, that this stage is the basic (elementary and secondary) one.

In addition to investigating the dynamics of inequality, this paper also analyzes economic growth. For the society as a whole, regimes that take longer to overcome inequality but show a permanently higher economic growth rate could be preferred to regimes with a fast reduction of inequality but low economic growth. A ranking of economic growth rates under different education regimes depends crucially on the values of the elasticities of human capital with respect to school-time investments and also with respect to human capital acquired during childhood.

For any fixed advanced education system, we show that private basic education leads to higher economic growth than public basic education in the long run. In order to gain some insight into the comparisons of economic growth rates, we calibrate our parameter values using educational data for OECD member countries. The results suggest that there is a clear tradeoff between economic growth and reduction of inequality, when we compare the completely private and the completely public regimes side by side.

Comparisons between the mixed educational regimes will depend not only on the aforementioned elasticities, but also on the interest rate. For the average interest rate of OECD countries (about 4% per year), the already mentioned growth vs. reduction of inequality tradeoff will persist. However, for a higher interest rate (6% or 7% per year, as in our numerical exercise), this tradeoff disappears: publicly funding basic education yields both a higher economic growth rate and a quicker reduction of inequality. This effect can only be appreciated due to the consideration of physical capital in our model.[2]

In the following section, we will describe the economic model for all of the four educational regimes. Section 3 contains the long-term theoretical results and their interpretations. In Section 4 we present the results from our calibration and alternative numerical exercises. Finally, Section 5 takes stock of the implications of our analysis, besides considering its limitations and possible extensions.

2 The model

Consider an economy populated by overlapping generations of individuals who live for four periods of fifteen years: childhood (c), youth (y), adulthood (a) and old age (o). They can have either one of two labels (each associated with a specific family name, so that an individual necessarily carries on the label of his parent), A and B, to be referred to as groups. Individuals are

[2]Cf. de la Croix and Doepke (2004).

indexed by $i \in \{A, B\}$ and their life period $j \in \{c, y, a, o\}$. At each period $t \in \{0, 1, 2, ...\}$, individual i, in her j-th period of life, where $j \in \{y, a\}$, has human capital level $h_t^{j,i}$.

Regarding the life cycle, at period t, an i-adult works, saves, gives birth to n_t^i children, purchases consumption goods for the household, and has an educational expenditure in accordance with one of the four educational regimes to be explained later on in this section. At $t+1$, that adult becomes old and her children become young. She spends past savings on household consumption, and, depending on the educational regime in place, may also have to cover her young kids' advanced education expenses.

At the beginning of time, for each $i \in \{A, B\}$, there are $P_0^{a,i} > 0$ i-adults, and they decide to have n_0^i children (also labeled i). In order to ensure coexistence of the four life periods from period 1 onward, society must be populated additionally with a new cohort of $P_1^{a,i} > 0$ adults in period 1. In order to ease calculations, we assume that $P_1^{a,i}/P_0^{a,i} = \sqrt{n_0^i}, \forall\, i \in \{A, B\}$. We assume also that individuals are endowed with human capital in such a way that $h_0^{a,B} = h_1^{a,B} \geq h_0^{a,A} = h_1^{a,A} > 0$.

The law of motion for the adult population is:

$$P_{t+2}^{a,i} = n_t^i P_t^{a,i}, \; \forall\, t \geq 0. \tag{1}$$

Adults make all the decisions. An i-adult at t cares about the number of children, n_t^i, current household consumption, $c_t^{a,i}$, future household consumption, $c_{t+1}^{o,i}$, and the human capital that her offspring will produce during the education phase, $h_{t+2}^{a,i}$. Her utility function is given by

$$u(c_t^{a,i}, c_{t+1}^{o,i}, n_t^i, h_{t+2}^{a,i}) = \ln c_t^{a,i} + \rho \ln c_{t+1}^{o,i} + \gamma \ln \left(n_t^i h_{t+2}^{a,i} \right), \tag{2}$$

where the parameter $\rho > 0$ is the psychological discount factor and $\gamma > 0$ is the altruism factor. Note, by the very functional form of u, that $h_2^{a,i}, h_3^{a,i}, ...$ will necessarily also be positive in equilibrium.

Adults at t work and earn a wage rate, per fraction of the period worked and per unit of their human capital, equal to w_t. To cover their expenses at $t+1$, when they will be retired, they need to save. The savings of adults at t are used as physical capital by the firm at $t+1$, after which it will pay back the rental rate r_{t+1}.

The education phase occurs during the first two stages of life: childhood and youth. During this phase, individuals only study. Children attend basic schooling (from kindergarten up to high school) and young individuals attend advanced education (college and graduate school). These two stages have distinct features. We assume that each stage of the education phase has a specific human-capital technology.

For an individual born at period t, the human capital available at the beginning of her youth period, $h_{t+1}^{y,i}$, is the output of f^c, the human capital production function during childhood. This function takes as inputs parental human capital, $h_t^{a,i}$, investments in basic education (amount of time spent at school), $e_t^{c,i}$, and the average quality of teachers as measured by their human capital levels, $\bar{h}_t$ (we assume that all children and young individuals attend each teacher's classes for the same amount of time). Let $\eta_1^c, \eta_2^c, \eta_3^c \in (0,1)$ denote the elasticities of the human capital produced during childhood with respect to parental human capital, to the school-time investment in basic education, and to teacher quality, respectively. The parameter $\mu^c > 0$ denotes a multiplicative constant of the human capital produced during childhood:

$$h_{t+1}^{y,i} = f^c\left(h_t^{a,i}, e_t^{c,i}, \bar{h}_t\right) = \mu^c \left(h_t^{a,i}\right)^{\eta_1^c} \left(e_t^{c,i}\right)^{\eta_2^c} \left(\bar{h}_t\right)^{\eta_3^c}. \tag{3}$$

This function of human capital produced during childhood is a particular case of the human capital formation technology used by Cunha and Heckman (2007). We use this particular case because it allows for better comparability between this and other works that address economic growth through human capital accumulation, for instance, de la Croix and Doepke (2004).[3]

Again for an individual born at t, the human capital available at the beginning of adulthood, $h_{t+2}^{a,i}$, is the output of f^y, the human capital production function during youth. This function takes as inputs the human capital produced during childhood, $h_{t+1}^{y,i}$, the school-time investment in advanced

[3]The only difference between our technology of children's human-capital formation and that of de la Croix and Doepke (2004) is the non-inclusion of an additive constant to investment in education. By not including this constant, we allow for the existence of equilibria in which inequality remains bounded away from 0 when education is private.

education, $e^{y,i}_{t+1}$, and the quality of teachers, $\bar{h}_{t+1}$. The human capital acquired during childhood is required for acquiring new knowledge in advanced education. Moreover, childhood education sets the stage for education during youth. Let $\eta^y_1, \eta^y_2, \eta^y_3 \in (0,1)$ denote the elasticities of the human capital produced during youth with respect to the human capital produced during childhood, to the school-time investment in advanced education, and to teacher quality, respectively. Let $\mu^y > 0$ denote a multiplicative constant in the human capital technology of the young:

$$h^{a,i}_{t+2} = f^y\left(h^{y,i}_{t+1}, e^{y,i}_{t+1}, \bar{h}_{t+1}\right) = \mu^y \left(h^{y,i}_{t+1}\right)^{\eta^y_1} \left(e^{y,i}_{t+1}\right)^{\eta^y_2} \left(\bar{h}_{t+1}\right)^{\eta^y_3}. \tag{4}$$

The human capital production function during youth, f^y, has two special features, as highlighted by Cunha and Heckman (2007): self-productivity and dynamic complementarity.[4] Similar specifications have been used in the literature that investigates the role of the hierarchical nature of education in human capital production (see the "special case" in Arcalean and Schiopu (2010) and Abington and Blankenau (2013)). By plugging (3) in (4), we find the human capital available for an adult that was born in period t, which is given below. To simplify, let μ denote the aggregate multiplicative constant $\mu^y (\mu^c)^{\eta^y_1}$:

$$h^{a,i}_{t+2} = \mu \left(h^{a,i}_{t}\right)^{\eta^c_1\eta^y_1} \left(e^{c,i}_{t}\right)^{\eta^c_2\eta^y_1} \left(\bar{h}_{t}\right)^{\eta^c_3\eta^y_1} \left(e^{y,i}_{t+1}\right)^{\eta^y_2} \left(\bar{h}_{t+1}\right)^{\eta^y_3}, \tag{5}$$

so that the combined parameter $\eta^c_2\eta^y_1 < 1$ is the elasticity of the human capital available at the beginning of adulthood with respect to the time investment in basic education. For simplicity, from this point on, $\eta^c_2\eta^y_1$ and η^y_2 will be called, respectively, *adults' basic education elasticity* and *adults' advanced education elasticity*. In order to generate convergence, we assume that, for fixed investment levels $e^{c,i}_t$ and $e^{y,i}_{t+1}$, both $f^c\left(h^{a,i}_t, e^{c,i}_t, \bar{h}_t\right)$ and $f^y\left(h^{y,i}_{t+1}, e^{c,i}_{t+1}, \bar{h}_{t+1}\right)$ present constant returns to scale, as in Tamura (1991) and de la Croix and Doepke (2004).

Assumption 1. $\eta^c_1 + \eta^c_3 = 1$ and $\eta^y_1 + \eta^y_3 = 1$.

[4] Self-productivity arises when human capital produced during the previous period increases the human capital produced during the current one. Dynamic complementarity means that human capital accumulated during the previous period increases the productivity of educational investments in the current one. This implies that investment in the schooling of children boosts investments in advanced education.

Educational regimes can be either private or public. For simplicity, we assume that different regimes cannot coexist at the same educational stage, so that there are four possible educational regimes: basic and advanced private education; basic and advanced public education; private basic and public advanced education; and public basic and private advanced education.

The following assumption will have the primary purpose of ensuring existence of equilibria under the completely private regime, as will be seen shortly.

Assumption 2. $\eta_2^c \eta_1^y + \eta_2^y < 1$.

Both of these assumptions will be taken for granted throughout this work.

2.1 Consumption good and the education markets

There are two markets in our economy: the education market and the consumption good market. There is free mobility of labor between them, so adults can work in both markets. At any period $t \in \{0, 1, 2, ...\}$, a single firm produces the consumption good, using physical capital and labor as inputs. Let K_t^d be the demand for physical capital and L_t^d, the demand for efficiency units of labor. The production technology has constant returns to scale:

$$Y_t = F(K_t^d, L_t^d) = L_t^d f(\kappa_t^d),$$

where $f(\cdot) = F(\cdot, 1)$ and $\kappa_t^d := K_t^d / L_t^d$ is the physical capital-labor ratio. The firm chooses the quantity of physical capital and effective labor to maximize its profit, π_t. Let r_t be the rental rate of physical capital and w_t, the wage per unit of human capital, so that

$$\pi_t = L_t^d f(\kappa_t^d) - r_t K_t^d - w_t L_t^d.$$

The first-order conditions of the firm's problem are:

$$f'(\kappa_t) = r_t, \tag{6}$$

$$f(\kappa_t) - f'(\kappa_t)\kappa_t = w_t. \tag{7}$$

At period 0, the stock of physical capital of the economy is set at a level $K_0 > 0$. We assume that this economy is small and open, with perfect mobility of physical capital, so that, from period 1 onward, the firm can freely borrow from abroad physical capital, to be readily available for production. For all $t \in \{0, 1, ...\}$, let S_t^x stand for external savings at t. Total domestic savings at t are composed of the total savings of adults at the end of period t, $P_t^{a,A} s_t^A + P_t^{a,B} s_t^B$. For simplicity, we assume that physical capital depreciates completely at every period. Thus,

$$K_{t+1} = P_t^{a,A} s_t^A + P_t^{a,B} s_t^B - S_t^x. \tag{8}$$

We assume that the world interest rate is constant over time, $r > 0$. This assumption is often used in the literature investigating human capital accumulation (see, for instance, Galor and Zeira, 1993; Abington and Blankenau, 2013). The free mobility of physical capital guarantees that the domestic interest rate equals the world interest rate:

$$r_t = r.$$

This and the first-order conditions of the firm's problem imply that the physical capital-labor ratio κ_t and the wage per human capital unit, w, are constant over time.

For simplicity, as in de la Croix and Doepke (2004), we assume that there must be an identical share of teachers in the A- and in the B-populations, so the average human capital of teachers equals the average human capital of adults:

$$\bar{h}_t = \bar{h}_t^a := \frac{P_t^{a,A} h_t^{a,A} + P_t^{a,B} h_t^{a,B}}{P_t^{a,A} + P_t^{a,B}}. \tag{9}$$

Adults from each group offer their labor to the firm. However, the available amount of time for work in goods production is limited by two factors. First, to raise each child, adults spend an exogenous, and invariant to population group, fraction $\phi \in (0, 1)$ of their time. Therefore, the time spent with children, ϕn_t^i, reduces the amount of time available for work. Besides that, as adults can also work as teachers, the time spent teaching children, $\sum_{i\in\{A,B\}} P_t^{a,i} e_t^{c,i} n_t^i$, and the young, $\sum_{i\in\{A,B\}} P_{t-1}^{a,i} e_t^{y,i} n_{t-1}^i$, is subtracted from aggregate labor supply for goods production. Thus, the effective (which considers quality of labor) labor supply available to the firm, L_t, is given as:

$$\begin{aligned} L_t = P_t^{a,A} &\left(\left(1-\phi n_t^A\right) h_t^{a,A} - e_t^{c,A} n_t^A \bar{h}_t^a \right) - P_{t-1}^{a,A} e_t^{y,A} n_{t-1}^A \bar{h}_t^a \\ + P_t^{a,B} &\left(\left(1-\phi n_t^B\right) h_t^{a,B} - e_t^{c,B} n_t^B \bar{h}_t \right) - P_{t-1}^{a,B} e_t^{y,B} n_{t-1}^B \bar{h}_t^a. \end{aligned} \tag{10}$$

Since there is free mobility of labor, the wage per unit of human capital, w, is the same across markets.

2.2 Basic and advanced private education

If both the basic and the advanced educational stages are private (from now on called the private-private regime), parents pay for the education of each child in both periods. The expenditure with basic education of the offspring of an i-individual who is an adult at t is $w_t n_t^i e_t^{c,i} \bar{h}_t^a$, while the expenditure with their advanced education is $w_{t+1} n_t^i e_{t+1}^{y,i} \bar{h}_{t+1}^a$. The budget constraints she faces are:

$$c_t^{a,i} + s_t^i = w_t h_t^{a,i}(1-\phi n_t^i) - w_t n_t^i e_t^{c,i} \bar{h}_t^a, \tag{11}$$

and

$$c_{t+1}^{o,i} = (1+r_{t+1}) s_t^i - w_{t+1} n_t^i e_{t+1}^{y,i} \bar{h}_{t+1}^a. \tag{12}$$

Adults face one more constraint, regarding the number of children to whom they decide to give birth. As already discussed, parents devote some amount of their time to take care of each of their children. Thus, when they decide how many children they want, they must take into account that the total amount of time allocated to take care of children cannot exceed their time endowment:

$$1 - \phi n_t^i \geq 0. \tag{13}$$

Definition. Given initial endowments of human capital of adults $\left(h_0^{a,i}, h_1^{a,i}\right)_{i\in\{A,B\}}$, initial population size $\left(P_0^{a,i}, P_1^{a,i}\right)_{i\in\{A,B\}}$ and an initial stock of physical capital K_0, an *equilibrium under the private-private educational regime* consists of sequences (for t varying in $\{0, 1, ...\}$) of prices (r_t, w_t), aggregate quantities $\left(L_t, K_{t+1}, \bar{h}_t^a\right)$, human capital levels $\left(h_t^{y,i}, h_t^{a,i}\right)_{i\in\{A,B\}}$, population sizes $\left(P_t^{a,i}\right)_{i\in\{A,B\}}$, and decision rules $\left(c_t^{a,i}, c_{t+1}^{o,i}, s_t^i, n_t^i, e_t^{c,i}, e_{t+1}^{y,i}\right)_{i\in\{A,B\}}$ such that, for all $t \in \{0, 1, ...\}$ and $i \in \{A, B\}$:

1 - household decisions $c_t^{a,i}$, $c_{t+1}^{o,i}$, s_t^i, n_t^i, $e_t^{c,i}$ and $e_{t+1}^{y,i}$ maximize utility function (2) subject to the constraints (11), (12), (13) and (5);

2 - the firm chooses K_t^d and L_t^d to maximize profits, i.e., (6) and (7) hold;

3 - prices w_t, r_t are such that markets clear, i.e., $K_t^d = K_t$ and $L_t^d = L_t$, where K_t and L_t are given by (8) and (10);

4 - population evolves according to (1) and $P_1^{a,i} = \sqrt{n_0^i} P_0^{a,i}$; and

5 - the aggregate variable $\bar{h}_t^a$ is given by (9).

Let $x_t^{a,i}$ denote the relative i-adult human capital at t, and g_t, the growth factor of the average human capital of adults from t to $t+1$:

$$x_t^i = \frac{h_t^{a,i}}{\bar{h}_t^a}, \tag{14}$$

$$g_t = \frac{\bar{h}_{t+1}^a}{\bar{h}_t^a}. \tag{15}$$

It should be noted for future reference that, since $0 < h_0^{a,A} \leq h_0^{a,B}$, we have $0 < x_0^A \leq 1$, and, since $h_1^{a,A} = h_0^{a,A}$ and $h_1^{a,B} = h_0^{a,B}$, we also have $x_1^A = x_0^A$ and $x_1^B = x_0^B$.

The first-order conditions imply:

$$n = \frac{\gamma \left(1 - \eta_2^y - \eta_2^c \eta_1^y\right)}{\phi(1+\rho+\gamma)}, \tag{16}$$

$$s_t^i = \frac{\left(\rho + \gamma \eta_2^y\right) w h_t^{a,i}}{1+\rho+\gamma}, \tag{17}$$

$$e_t^{c,i} = \frac{\eta_2^c \eta_1^y \phi x_t^i}{1 - \eta_2^y - \eta_2^c \eta_1^y}, \tag{18}$$

$$e_{t+1}^{y,i} = \frac{\eta_2^y \left(1+r\right) \phi x_t^i}{g_t \left(1 - \eta_2^y - \eta_2^c \eta_1^y\right)}. \tag{19}$$

It is clear from (16) that constraint (13) is satisfied. Assumption 2 ensures that $\eta_2^c \eta_1^y + \eta_2^y < 1$ and, therefore, that fertility and educational investments decisions in the private-private educational regime are positive and well-defined. This fertility decision is constant over time and is group-invariant. The number of children is negatively related to both adults' basic and advanced education elasticities, $\eta_2^c \eta_1^y$ and η_2^y.

Investments in the education of children and young individuals increase with parental relative human capital. This means that richer (i.e., more skilled) parents will invest more in the education of their offspring. Moreover, note that, in equilibrium, both basic and advanced education school-time investment, (18) and (19), increase with adults' basic and advanced education elasticities. Since parents want to invest more in basic and advanced schooling when these elasticities are high, they will have fewer children, so that they are able to afford more education for them.

2.3 Basic and advanced public education

Since there must be an identical share of teachers in the A- and in the B-populations, the average human capital of adults (equivalently, of teachers) has external effects on the production of human capital. This externality may suggest the possibility of government interventions. In the present regime, the government chooses how much to invest at each educational stage knowing that this education policy affects the average human capital, thereby partially internalizing this spillover effect.

In an exclusively public educational regime (also called the public-public regime), schooling is provided only publicly, and this is the government's only role. To this end, it levies a proportional income tax on adults, ν_t, and uses the proceeds to finance education in both periods. The budget constraints of an i-individual who is an adult at t under the completely public regime are:

$$c_t^{a,i} + s_t^i = (1 - \nu_t)\, w_t h_t^{a,i} \left(1 - \phi n_t^i\right) \tag{20}$$

and

$$c_{t+1}^{o,i} = (1 + r_{t+1})\, s_t^i, \tag{21}$$

since seniors are not taxed and do not pay for the education of the young. The savings of adults are used for household consumption when they are old:

The central authority does not discriminate in any direction individuals by their group, so that its school-time investment decisions, both for basic and advanced education, are group-invariant. However, educational expenditures can vary between educational stages. The human capital available to an i-adult at $t+2$ in an exclusively public regime is

$$h_{t+2}^{a,i} = \mu \left(h_t^{a,i}\right)^{\eta_1^c \eta_1^y} \left(\bar{e}_t^c\right)^{\eta_2^c \eta_1^y} \left(\bar{h}_t^a\right)^{\eta_3^c \eta_1^y} \left(\bar{e}_{t+1}^y\right)^{\eta_2^y} \left(\bar{h}_{t+1}^a\right)^{\eta_3^y} \tag{22}$$

Government terms last for only one period. During its term, the public authority chooses the level of education at each stage to maximize the welfare of adults and senior citizens (that is, the government relies on parents to take the well-being of their offspring into account):

$$P_t^{a,A} u(c_t^{a,A}, c_{t+1}^{o,A}, n_t^A, h_{t+2}^{a,A}) + P_t^{a,B} u(c_t^{a,B}, c_{t+1}^{o,B}, n_t^B, h_{t+2}^{a,B}) \tag{23}$$

$$+ P_{t-1}^{a,A} u(c_{t-1}^{a,A}, c_t^{o,A}, n_{t-1}^A, h_{t+1}^{a,A}) + P_{t-1}^{a,B} u(c_{t-1}^{a,B}, c_t^{o,B}, n_{t-1}^B, h_{t+1}^{a,B}),$$

where $P_{-1}^{a,A} := 0$ and $P_{-1}^{a,B} := 0$ (so that the second line above vanishes if one plugs in $t = 0$).

It is worth noting that decisions made by current period seniors in the past are included in the current period welfare function. However, as utility is logarithmic, these chosen variables enter the welfare function in an additive fashion, whence do not affect the current government's decision.

The government must always run a balanced budget, so that total expenditure on education at any period t must equal total tax revenue at t. Therefore, in its maximization problem, the government must satisfy the following budget constraint:

$$\begin{aligned} & w_t \bar{e}_t^c \bar{h}_t^a \left(P_t^{a,A} n_t^A + P_t^{a,B} n_t^B \right) + w_t \bar{e}_t^y \bar{h}_t^a \left(P_{t-1}^{a,A} n_{t-1}^A + P_{t-1}^{a,B} n_{t-1}^B \right) \\ & = w_t \nu_t \left(P_t^{a,A} h_t^{a,A} \left(1 - \phi n_t^A\right) + P_t^{a,B} h_t^{a,B} \left(1 - \phi n_t^B\right) \right) \end{aligned} \tag{24}$$

Definition. Given initial endowments of human capital $\left(h_0^{a,i}, h_1^{a,i}\right)_{i \in \{A,B\}}$, initial population size $\left(P_0^{a,i}, P_1^{a,i}\right)_{i \in \{A,B\}}$ and an initial stock of physical capital K_0, an *equilibrium under the public-public educational regime* consists of sequences (for t varying in $\{0, 1, ...\}$) of prices (r_t, w_t), aggregate quantities $\left(L_t, K_{t+1}, \bar{h}_t^a\right)$, human capital levels $\left(h_t^{y,i}, h_t^{a,i}\right)_{i \in \{A,B\}}$, population sizes $\left(P_{t+1}^{a,i}\right)_{i \in \{A,B\}}$, decision rules $\left(c_t^{a,i}, c_{t+1}^{o,i}, s_t^i, n_t^i\right)_{i \in \{A,B\}}$, and policy variables $\left(\bar{e}_t^c, \bar{e}_{t+1}^y, \nu_t\right)$ such that, for all $t \in \{0, 1, ...\}$ and $i \in \{A, B\}$:

1 - household decisions $c_t^{a,i}$, $c_{t+1}^{o,i}$, s_t^i and n_t^i maximize adults' utility function (2) subject to the constraints (20), (21), (13) and (22);

2 - given household decisions at t and $t-1$ (unless $t = 0$, in which case only decisions at t must be given) and (22), the government's choices of $\bar{e}_t^c$ and $\bar{e}_{t+1}^y$ maximize (23) subject to (24);

3 - the firm chooses K_t^d and L_t^d to maximize profits, i.e., (6) and (7) hold;

4 - prices w_t, r_t are such that markets clear, i.e., $K_t^d = K_t$ and $L_t^d = L_t$, where K_t and L_t are given by (8) and (10);

5 - population evolves according to (1) and $P_1^{a,i} = \sqrt{n_0^i} P_0^{a,i}$; and

6 - the aggregate variable $\bar{h}_t^a$ is given by (9).

The solution of the household decision problem is interior, and the first-order conditions imply:

$$n = \frac{\gamma}{\phi\left(1+\rho+\gamma\right)}, \tag{25}$$

$$s_t^i = \frac{\rho\left(1-\nu_t\right)wh_t^{a,i}}{1+\rho+\gamma}.$$

Again, the fertility decision satisfies $n \leq 1/\phi$, is constant over time and is group-invariant. Contrary to the private-private regime, in the public-public regime, fertility does not depend on the elasticities of the human capital production function. When the government funds the education system, parents choose to have more children than in a completely private regime, as can be seen in (16) and (25). This happens because parents do not pay for education directly. Given the choice of number of children, the government budget constraint (24) is simplified to

$$\nu_t = \frac{\gamma}{\left(1-\rho\right)\phi}\left(\bar{e}_t^c + \frac{\bar{e}_t^y}{\left(\frac{\gamma}{\phi(1+\rho+\gamma)}\right)^{\frac{1}{2}}}\right). \tag{26}$$

The government maximizes the welfare of adults and seniors, (23), by choosing the level of basic and advanced education provided subject to (26). By doing so, it is aware that its choices can affect future variables and, therefore, the intertemporal utility of current adults. The first-order condition of the welfare maximization problem gives:

$$\bar{e}_t^c = \frac{\eta_2^c\eta_1^y\left(1+\rho\right)\phi\left(\frac{\gamma}{(1+\rho+\gamma)\phi}\right)^{\frac{1}{2}}}{\left(1+\rho+\gamma\left(\eta_2^c\eta_1^y+\left(1-\eta_1^y\right)\eta_2^y\right)\right)\left(\frac{\gamma}{(1+\rho+\gamma)\phi}\right)^{\frac{1}{2}}+\gamma\eta_2^y}, \tag{27}$$

$$\bar{e}_t^y = \frac{\left(\frac{\gamma}{(1+\rho+\gamma)\phi}\right)^{\frac{1}{2}}\phi\left(1+\rho\right)\left(\left(1-\eta_1^y\right)\eta_2^y\left(\frac{\gamma}{(1+\rho+\gamma)\phi}\right)^{\frac{1}{2}}+\eta_2^y\right)}{\left(1+\rho+\gamma\left(\eta_2^c\eta_1^y+\left(1-\eta_1^y\right)\eta_2^y\right)\right)\left(\frac{\gamma}{(1+\rho+\gamma)\phi}\right)^{\frac{1}{2}}+\gamma\eta_2^y}. \tag{28}$$

When the government raises the ongoing investment in the education of each young individual, $\bar{e}_t^y$, it boosts the average human capital of next period adults, $\bar{h}_{t+1}^a$. Consequently, in the future, teachers will be more qualified, thereby increasing the human capital produced by the offspring of this period's adults, $h_{t+2}^{a,i}$. Thus, an increase of current investment in the education of the young will affect the utility of current adults positively.

By plugging (27) and (28) in (26), we obtain the labor income tax:

$$\nu_t = \frac{\gamma\left(\left(\frac{\gamma}{(1+\rho+\gamma)\phi}\right)^{\frac{1}{2}}\left(\eta_2^c\eta_1^y + \left(1-\eta_1^y\right)\eta_2^y\right) + \eta_2^y\right)}{\left(1+\rho+\gamma\left(\eta_2^c\eta_1^y + \left(1-\eta_1^y\right)\eta_2^y\right)\right)\left(\frac{\gamma}{(1+\rho+\gamma)\phi}\right)^{\frac{1}{2}} + \gamma\eta_2^y}.$$

It is straightforward to check that $\partial\bar{e}_t^y/\partial(\eta_2^c\eta_1^y) < 0$ and $\partial\bar{e}_t^c/\partial\eta_2^y < 0$, that is, the greater the adults' basic (advanced) education elasticity, the lower will the investments in advanced (basic) education be. As expected, adults' human capital elasticity with respect to one stage of education has positive effects on educational investments at that stage: $\partial\bar{e}_t^c/\partial(\eta_2^c\eta_1^y) > 0$ and $\partial\bar{e}_t^y/\partial\eta_2^y > 0$.

2.4 Private basic and public advanced education

Unlike the two educational regimes previously described, basic and advanced school systems could be of different types. Here, we consider a regime where parents directly pay for the basic education of their children, but pay for advanced education of the young through taxes. From now on this educational regime will be called the private-public regime.

To fund advanced public education, the government levies a proportional labor income tax on adults. The budget constraints of an i-individual who is an adult at t are given below:

$$c_t^{a,i} + s_t^i = \left(1-\nu_t\right)w_t h_t^{a,i}(1-\phi n_t^i) - w_t n_t^i e_t^{c,i}\bar{h}_t^a, \tag{29}$$

$$c_{t+1}^{o,i} = (1+r_{t+1})s_t^i. \tag{30}$$

Once more, adults make all the private decisions. In order to maximize the utility function, adults choose the household consumption in the current and the next period, as well as the number of children they will have and the investments they will make in childhood education. Under this regime, investments in basic education can vary across groups, while investments in advanced education, which is funded by the government, cannot:

$$h_{t+2}^{a,i} = \mu \left(h_t^{a,i}\right)^{\eta_1^c \eta_1^y} \left(e_t^{c,i}\right)^{\eta_2^c \eta_1^y} \left(\bar{h}_t\right)^{\eta_3^c \eta_1^y} \left(\bar{e}_{t+1}^y\right)^{\eta_2^y} \left(\bar{h}_{t+1}\right)^{\eta_3^y}. \tag{31}$$

The government maximizes the welfare of adults and seniors, (23), by choosing how much to invest in the education of the young. Adults will pay the necessary amount of income tax to keep the government budget constraint satisfied:

$$w_t \bar{e}_t^y \bar{h}_t \left(P_{t-1}^{a,A} n_{t-1}^A + P_{t-1}^{a,B} n_{t-1}^B\right) = w_t \nu_t \left(P_t^{a,A} h_t^{a,A} \left(1 - \phi n_t^A\right) + P_t^{a,B} h_t^{a,B} \left(1 - \phi n_t^B\right)\right). \tag{32}$$

Definition. Given initial endowments of human capital $\left(h_0^{a,i}, h_1^{a,i}\right)_{i\in\{A,B\}}$, initial population size $\left(P_0^{a,i}, P_1^{a,i}\right)_{i\in\{A,B\}}$ and an initial stock of physical capital K_0, an *equilibrium under the private-public educational regime* consists of sequences (for t varying in $\{0, 1, ...\}$) of prices (r_t, w_t), aggregate quantities $\left(L_t, K_{t+1}, \bar{h}_t^a\right)$, human capital levels $\left(h_t^{y,i}, h_t^{a,i}\right)_{i\in\{A,B\}}$, population sizes $\left(P_{t+1}^{a,i}\right)_{i\in\{A,B\}}$, household decisions $\left(c_t^{a,i}, c_{t+1}^{o,i}, s_t^i, n_t^i, e_t^{c,i}\right)_{i\in\{A,B\}}$, and policy variables $\left(\bar{e}_{t+1}^y, \nu_t\right)$ such that, for all $t \in \{0, 1, ...\}$ and $i \in \{A, B\}$:

1 - household decisions $c_t^{a,i}$, $c_{t+1}^{o,i}$, s_t^i, n_t^i and $e_t^{c,i}$ maximize utility function (2) subject to the constraints (29), (30), (13) and (31);

2 - given household decisions at t and $t-1$ (unless $t = 0$, in which case only decisions at t must be given) and (22), the government's choice $\bar{e}_t^y$ maximizes (23) subject to (32);

3 - the firm chooses K_t^d and L_t^d to maximize profits, i.e., (6) and (7) hold;

4 - prices w_t, r_t are such that markets clear, i.e., $K_t^d = K_t$ and $L_t^d = L_t$, where K_t and L_t are given by (8) and (10);

5 - population evolves according to (1) and $P_1^{a,i} = \sqrt{n_0^i} P_0^{a,i}$; and

6 - the aggregate variable $\bar{h}_t^a$ is given by (9).

The first-order conditions of an i-adult's problem are:

$$n = \frac{\gamma\left(1 - \eta_2^c \eta_1^y\right)}{\phi\left(1 + \rho + \gamma\right)}, \tag{33}$$

$$s_t^i = \frac{\rho\left(1 - \nu_t\right) w h_t^{a,i}}{1 + \rho + \gamma},$$

$$e_t^{c,i} = \frac{\eta_2^c \eta_1^y \left(1 - \nu_t\right) \phi x_t^i}{1 - \eta_2^c \eta_1^y}. \tag{34}$$

The fertility choice once again satisfies $n \leq 1/\phi$, is constant over time and across groups. However, differently from the public-public regime, in the private-public regime, adults' basic education elasticity $\eta_2^c \eta_1^y$ has an adverse effect on fertility choice. This elasticity has also a positive impact on the private choice of investment in basic education: the more productive basic education is, the more time parents want their children to spend at school. Adults' advanced education elasticity η_2^y has no effect on decision choices in this mixed model. Given (33), the government budget constraint (32) is simplified to:

$$\bar{e}_t^y = \frac{\phi\left(1 + \rho + \gamma \eta_2^c \eta_1^y\right)\left(\frac{\gamma\left(1 - \eta_2^c \eta_1^y\right)}{\phi\left(1 + \rho + \gamma\right)}\right)^{\frac{1}{2}}}{\gamma\left(1 - \eta_2^c \eta_1^y\right)} \nu_t. \tag{35}$$

Given the choices for the consumer problem, the public authority maximizes the welfare of adults and seniors, (23), by choosing how much to invest in the education of the young subject to (35). The first-order condition of the problem of the government is:

$$\bar{e}_t^y = \frac{\left(\left(\frac{\gamma\left(1 - \eta_2^c \eta_1^y\right)}{\phi\left(1 + \rho + \gamma\right)}\right)^{\frac{1}{2}}\left(1 - \eta_1^y\right)\eta_2^y + \eta_2^y\right)\phi\left(1 + \rho + \gamma \eta_2^c \eta_1^y\right)\left(\frac{\gamma\left(1 - \eta_2^c \eta_1^y\right)}{\phi\left(1 + \rho + \gamma\right)}\right)^{\frac{1}{2}}}{\left(1 - \eta_2^c \eta_1^y\right)\left(\left(1 + \rho + \gamma\left(\eta_2^c \eta_1^y + \left(1 - \eta_1^y\right)\eta_2^y\right)\right)\left(\frac{\gamma\left(1 - \eta_2^c \eta_1^y\right)}{\phi\left(1 + \rho + \gamma\right)}\right)^{\frac{1}{2}} + \gamma \eta_2^y\right)}.$$

Given this level of education, the labor income tax is set to satisfy the government budget constraint:

$$\nu_t = \frac{\gamma\left(\left(\frac{\gamma\left(1-\eta_2^c\eta_1^y\right)}{\phi(1+\rho+\gamma)}\right)^{\frac{1}{2}}\left(1-\eta_1^y\right)\eta_2^y+\eta_2^y\right)}{\left(1+\rho+\gamma\left(\eta_2^c\eta_1^y+\left(1-\eta_1^y\right)\eta_2^y\right)\right)\left(\frac{\gamma\left(1-\eta_2^c\eta_1^y\right)}{\phi(1+\rho+\gamma)}\right)^{\frac{1}{2}}+\gamma\eta_2^y}. \tag{36}$$

Finally, basic education investments may be found by replacing the income tax rate (36) in (34):

$$e_t^{c,i} = \frac{\left(1+\rho+\gamma\eta_2^c\eta_1^y\right)\left(\frac{\gamma\left(1-\eta_2^c\eta_1^y\right)}{\phi(1+\rho+\gamma)}\right)^{\frac{1}{2}}\eta_2^c\eta_1^y\phi x_t^i}{\left(1-\eta_2^c\eta_1^y\right)\left(\left(1+\rho+\gamma\left(\eta_2^c\eta_1^y+\left(1-\eta_1^y\right)\eta_2^y\right)\right)\left(\frac{\gamma\left(1-\eta_2^c\eta_1^y\right)}{\phi(1+\rho+\gamma)}\right)^{\frac{1}{2}}+\gamma\eta_2^y\right)}.$$

2.5 Public basic and private advanced education

In this second mixed regime, basic education is public, while advanced education is private. For simplicity, this educational regime will be called the public-private regime. Under this regime, parents do not choose the level of school-time investment in the basic education of each of their children. Instead, the government will choose the same amount of education for all children in the economy. However, once their kids become young individuals, parents can choose how much to invest in their advanced education. To fund basic education, the government levies a proportional labor income tax on adults. The budget constraint of an i-individual who is an adult at t are given below:

$$c_t^{a,i} + s_t^i = \left(1-\nu_t\right) w_t h_t^{a,i}\left(1-\phi n_t^i\right), \tag{37}$$

$$c_{t+1}^{o,i} = (1+r_{t+1})s_t^i - w_{t+1} n_t^i e_{t+1}^{y,i}\bar{h}_{t+1}^a. \tag{38}$$

The inputs of the human capital production function should reflect the facts that all children will be served the same amount of education during the period of basic public education, but also that the choice of investing in advanced education is private:

$$h_{t+2}^{a,i} = \mu \left(h_t^{a,i}\right)^{\eta_1^c \eta_1^y} \left(\bar{e}_t^c\right)^{\eta_2^c \eta_1^y} \left(\bar{h}_t\right)^{\eta_3^c \eta_1^y} \left(e_{t+1}^{y,i}\right)^{\eta_2^y} \left(\bar{h}_{t+1}\right)^{\eta_3^y}. \tag{39}$$

The government maximizes the welfare of adults and seniors, (23), by choosing the level of investment in basic education for each child. Adults will pay the necessary amount of income tax to keep the government budget constraint satisfied:

$$w_t \bar{e}_t^c \bar{h}_t \left(P_t^{a,A} n_t^A + P_t^{a,B} n_t^B\right) = w_t \nu_t \left(P_t^{a,A} h_t^{a,A}\left(1 - \phi n_t^A\right) + P_t^{a,B} h_t^{a,B}\left(1 - \phi n_t^B\right)\right). \tag{40}$$

Definition. Given initial endowments of human capital $\left(h_0^{a,i}, h_1^{a,i}\right)_{i\in\{A,B\}}$, initial population size $\left(P_0^{a,i}, P_1^{a,i}\right)_{i\in\{A,B\}}$ and an initial stock of physical capital K_0, an *equilibrium under the public-private regime* consists of sequences of prices (for t varying in $\{0, 1, ...\}$) (r_t, w_t), aggregate quantities $\left(L_t, K_{t+1}, \bar{h}_t^a\right)$, human capital levels $\left(h_t^{y,i}, h_t^{a,i}\right)_{i\in\{A,B\}}$, population sizes $\left(P_{t+1}^{a,i}\right)_{i\in\{A,B\}}$, decision rules $\left(c_t^{a,i}, c_{t+1}^{o,i}, s_t^i, n_t^i, e_{t+1}^{y,i}\right)_{i\in\{A,B\}}$, and policy variables $(\bar{e}_t^c, \nu_t)$ such that, for all $t \in \{0, 1, ...\}$ and $i \in \{A, B\}$:

1 - household decisions $c_t^{a,i}$, $c_{t+1}^{o,i}$, s_t^i, n_t^i and $e_{t+1}^{y,i}$ maximize utility function (2) subject to constraints (37), (38), (13) and (39);

2 - given household decisions at t and $t-1$ (unless $t = 0$, in which case only decisions at t must be given) and (22), the government's choice $\bar{e}_t^c$ maximizes (23) subject to (40);

3 - the firm chooses K_t^d and L_t^d to maximize profits, i.e., (6) and (7) hold;

4 - prices w_t, r_t are such that markets clear, i.e., $K_t^d = K_t$ and $L_t^d = L_t$, where K_t and L_t are given by (8) and (10);

5 - population evolves according to (1) and $P_1^{a,i} = \sqrt{n_0^i} P_0^{a,i}$; and

6 - the aggregate variable $\bar{h}_t^a$ is given by (9).

The first-order conditions of an i-adult's problem are:

$$n=\frac{\gamma\left(1-\eta_2^y\right)}{\phi\left(1+\rho+\gamma\right)},\tag{41}$$

$$s_t^i=\frac{\left(\rho+\gamma\eta_2^y\right)\left(1-\nu_t\right)wh_t^{a,i}}{1+\rho+\gamma},$$

$$e_{t+1}^{y,i}=\frac{\phi\eta_2^y\left(1+r\right)\left(1-\nu_t\right)x_t^i}{g_t\left(1-\eta_2^y\right)}.\tag{42}$$

The fertility choice once again satisfies $n\leq 1/\phi$, is constant over time and across groups. Adults' advanced education elasticity, η_2^y, plays an important role in the choice of fertility and the level of investment in advanced education. The more productive advanced education is, the more parents invest in the education of their offspring in this stage. At the same time, a higher productivity of advanced education yields a lower fertility rate. In fact, these two effects could not go in the same direction, since adults must satisfy their budget constraints. Under this education regime, adults' basic education elasticity, $\eta_1^y\eta_2^c$, does not affect household decisions. Given the choice of quantity of children, the government budget constraint (40) is simplified to:

$$\nu_t=\frac{\bar{e}_t^c\gamma\left(1-\eta_2^y\right)}{\phi\left(1+\rho+\gamma\eta_2^y\right)}.\tag{43}$$

Given the choices for the consumer problem, the public authority maximizes the welfare of adults and seniors, (23), by choosing the level of investment in the basic education of children, subject to (43). The first-order condition of the problem of the government is:

$$\bar{e}_t^c=\frac{\phi\left(1+\rho+\gamma\eta_2^y\right)\eta_2^c\eta_1^y}{\left(1+\rho+\gamma\eta_2^y+\gamma\eta_2^c\eta_1^y\right)\left(1-\eta_2^y\right)}.\tag{44}$$

Therefore, the income tax rate is

$$\nu_t=\frac{\gamma\eta_2^c\eta_1^y}{1+\rho+\gamma\eta_2^y+\gamma\eta_2^c\eta_1^y}.\tag{45}$$

Note that the labor income tax increases with adults' basic education elasticity, while the opposite is true regarding adults' advanced education elasticity.

Using (42) and (45), we find the school-time investment in advanced education:

$$e_{t+1}^{y,i} = \frac{\phi\eta_2^y (1+r) x_t^i}{g_t (1-\eta_2^y)} \frac{1+\rho+\gamma\eta_2^y}{1+\rho+\eta_2^y\gamma+\gamma\eta_2^c\eta_1^y} \tag{46}$$

As can be noted from (16), (25), (33) and (41), fertility varies with educational regime. The completely public regime is the one under which parents have the most children, whereas the completely private regime is the one under which parents have the fewest children. This is due to a quality-quantity tradeoff: in a regime with at least one private educational stage, parents choose to have fewer children, so that they can invest more in each child. However, this tradeoff is inexistent in a regime where educational funding is entirely public, and school-time investments are decided by the central authority. These expressions, together with (18), (19), (34) and (42), immediately show how fertility and investment differentials among regimes hinge upon adults' basic and advanced education elasticities: that with respect to investment at any private educational stage negatively affects fertility, while positively affecting investment itself.

3 Equilibrium dynamics

The equilibrium dynamics can be expressed using two auxiliary variables only: the relative human capital of group A, x^A, and the growth factor of the average human capital, g. In fact, for instance under the private-private regime, the trajectory of g alongside the initial values $h_0^{a,A}$ and $h_0^{a,B}$ gives, through (15), $\bar{h}$. This, together with the trajectory of x^A plugged into (14), gives $h^{a,A}$ and $h^{a,B}$. Also, s^i, $e^{c,i}$ and $e^{y,i}$, for $i \in \{A, B\}$, will be determined through the first-order conditions (17), (18) and (19). Expression (16) and the initial values $P_0^{a,A}$ and $P_0^{a,B}$, as well as $P_1^{a,A}$ and $P_1^{a,B}$, gives the trajectories of $P^{a,A}$ and $P^{a,B}$ (both for even and odd time periods), so that (10) yields the trajectory of L. Since κ is determined by the constant international interest rate (as well as r and w), once L is determined, K is too. Finally, the consumption will come from the budget constraints (11), (12).

Moreover, since κ is constant, the growth factor of GDP is $Y_{t+1}/Y_t = K_{t+1}/K_t = L_{t+1}/L_t$. In Lemma 2, we will see that g is asymptotically equivalent to the growth factor of GDP per capita, so that, in the long run, economic growth is determined by the growth of average human capital.

As shown below, the x^A equilibrium path will not depend upon g, and this is true for any educational regime. Therefore, g will be determined by the x^A trajectory together with the initial condition $g_0 = \bar{h}_1^a/\bar{h}_0^a = 1$. Also, the dynamics for x^A will present only one steady state within the $(0,1]$ interval, which will be shown, under a reasonable hypothesis, to be the limit of x^A, regardless of the educational regime.

This model will not present a balanced growth path, in the sense of an equilibrium with constant g. However, as we shall see, as x^A converges to 1, also g will converge.

All the expressions in the following subsections hold for all $t \in \{0, 1, ...\}$.

Private-private regime:

In order to derive the relative human capital dynamics, we plug the educational investment choices (18) and (19) into (5) to obtain

$$\frac{x_{t+2}^i}{(x_t^i)^{z_{\text{priv-priv}}}} = J_{\text{priv-priv}} \left(\frac{1}{g_t}\right)^{\eta_1^y + \eta_2^y} \left(\frac{1}{g_{t+1}}\right), \tag{47}$$

where

$$z_{\text{priv-priv}} := (\eta_1^c + \eta_2^c)\,\eta_1^y + \eta_2^y, \tag{48}$$

and

$$J_{\text{priv-priv}} := \mu\left(\eta_2^c\eta_1^y\right)^{\eta_2^c\eta_1^y}\left(\eta_2^y\left(1+r\right)\right)^{\eta_2^y}\left(\frac{\phi}{1-\eta_2^c\eta_1^y-\eta_2^y}\right)^{\eta_2^c\eta_1^y+\eta_2^y}. \tag{49}$$

Due to i entering the left-hand side of (47) only, this expression provides a simple relationship between the relative human capital of the A- and B-populations:

$$\frac{x_{t+2}^{A}}{\left(x_{t}^{A}\right)^{z_{\text{priv-priv}}}}=\frac{x_{t+2}^{B}}{\left(x_{t}^{B}\right)^{z_{\text{priv-priv}}}}. \tag{50}$$

In order to simplify several of the following expressions, let us define the relative size of the adult A-population:

$$\zeta_{t}:=\frac{P_{t}^{a,A}}{P_{t}^{a,A}+P_{t}^{a,B}}. \tag{51}$$

In all educational regimes, the fertility choice n is constant over time and across groups, by (16), (25), (33) and (41). Therefore, the motion law for ζ_t is:

$$\zeta_{t+2}=\frac{P_{t+2}^{a,A}}{P_{t+2}^{a,A}+P_{t+2}^{a,B}}=\frac{P_{t}^{a,A}n}{P_{t}^{a,A}n+P_{t}^{a,B}n}=\frac{P_{t}^{a,A}}{P_{t}^{a,A}+P_{t}^{a,B}}=\zeta_{t}.$$

Since

$$\zeta_{1}=\frac{P_{1}^{a,A}}{P_{1}^{a,A}+P_{1}^{a,B}}=\frac{P_{0}^{a,A}\sqrt{n}}{P_{0}^{a,A}\sqrt{n}+P_{0}^{a,B}\sqrt{n}}=\frac{P_{0}^{a,A}}{P_{0}^{a,A}+P_{0}^{a,B}}=\zeta_{0},$$

the A-population relative size, ζ_t, is constant over time and equal to the initial exogenous relative size, ζ_0. Thus, we may write simply ζ from now on. Note, for future reference, that $\zeta=\zeta_0\in(0,1)$.

Another relationship between the relative human capital of the A- and B-populations is found by dividing both sides of (9) by $\bar{h}_t$ and using (51) and (14):

$$1=\zeta x_{t+2}^{A}+(1-\zeta)\,x_{t+2}^{B}. \tag{52}$$

Now, solving (50) for x_{t+2}^{B} and plugging it in (52), we find

$$x_{t+2}^{A}=\frac{1}{\zeta+(1-\zeta)\left(\frac{x_{t}^{B}}{x_{t}^{A}}\right)^{z_{\text{priv-priv}}}}. \tag{53}$$

Lagging (52) in two periods, solving it for x_t^B and plugging it in (53), we find the motion law of the A-population relative human capital:

$$x_{t+2}^{A} = \frac{1}{\zeta + (1-\zeta)\left(\frac{\frac{1}{x_t^A} - \zeta}{1-\zeta}\right)^{z_{\text{priv-priv}}}}, \tag{54}$$

To find the dynamics of the growth factor of the average human capital under the private-private educational regime, we substitute (54) in (47):

$$g_{t+1} \quad = J_{\text{priv-priv}} \left(\frac{1}{g_t}\right)^{\eta_1^y + \eta_2^y} \left\{\zeta \left(x_t^A\right)^{z_{\text{priv-priv}}} + (1-\zeta)\left(\frac{1-\zeta x_t^A}{1-\zeta}\right)^{z_{\text{priv-priv}}}\right\}, \tag{55}$$

Public-public regime:

Following the same procedure as above, we find the following dynamic system for an economy under the public-public educational regime:

$$x_{t+2}^{A} = \frac{1}{\zeta + (1-\zeta)\left(\frac{\frac{1}{x_t^A} - \zeta}{1-\zeta}\right)^{z_{\text{pub-pub}}}},$$

where

$$z_{\text{pub-pub}} := \eta_1^c \eta_1^y, \tag{56}$$

and

$$g_{t+1} = J_{\text{pub-pub}} \left(\frac{1}{g_t}\right)^{\eta_1^y} \left\{\zeta \left(x_t^A\right)^{z_{\text{pub-pub}}} + (1-\zeta)\left(\frac{1-\zeta x_t^A}{1-\zeta}\right)^{z_{\text{pub-pub}}}\right\}, \tag{57}$$

where

$$J_{\text{pub-pub}} := \quad \mu \left(\eta_2^c \eta_1^y\right)^{\eta_2^c \eta_1^y} \left(\eta_2^y \left(1 + (1-\eta_1^y)\sqrt{\frac{\gamma}{\phi(1+\rho+\gamma)}}\right)\right)^{\eta_2^y} \left(\frac{\frac{\phi}{1+\gamma\frac{\eta_2^c \eta_1^y}{1+\rho}}}{1+\gamma\eta_2^y \frac{1-\eta_1^y + \sqrt{\frac{\phi(1+\rho+\gamma)}{\gamma}}}{1+\rho+\gamma\eta_2^c\eta_1^y}}\right)^{\eta_2^c \eta_1^y + \eta_2^y} \tag{58}$$

Private-public regime:

The following dynamic system describes the economy under the private-public educational regime:

$$x_{t+2}^{A}=\frac{1}{\zeta+(1-\zeta)\left(\frac{\frac{1}{x_t^A}-\zeta}{1-\zeta}\right)^{z_{\text{priv-pub}}}},$$

where

$$z_{\text{priv-pub}}:=\eta_1^y\left(\eta_1^c+\eta_2^c\right), \tag{59}$$

and

$$g_{t+1}=J_{\text{priv-pub}}\left(\frac{1}{g_t}\right)^{\eta_1^y}\left\{\zeta\left(x_t^A\right)^{z_{\text{priv-pub}}}+(1-\zeta)\left(\frac{1-\zeta x_t^A}{1-\zeta}\right)^{z_{\text{priv-pub}}}\right\}, \tag{60}$$

where

$$J_{\text{priv-pub}}:=\mu\left(\eta_2^c\eta_1^y\right)^{\eta_2^c\eta_1^y}\left(\eta_2^y\left(1+(1-\eta_1^y)\sqrt{\frac{\gamma\left(1-\eta_2^c\eta_1^y\right)}{\phi\left(1+\rho+\gamma\right)}}\right)\right)^{\eta_2^y}\left(\frac{\frac{\phi}{1-\eta_2^c\eta_1^y}}{1+\gamma\eta_2^y\frac{1-\eta_1^y+\sqrt{\frac{\phi(1+\rho+\gamma)}{\gamma\left(1-\eta_2^c\eta_1^y\right)}}}{1+\rho+\gamma\eta_2^c\eta_1^y}}\right)^{\eta_2^c\eta_1^y+\eta_2^y}. \tag{61}$$

Public-private regime:

The following dynamic system describes the economy under the public-private educational regime:

$$x_{t+2}^{A}=\frac{1}{\zeta+(1-\zeta)\left(\frac{\frac{1}{x_t^A}-\zeta}{1-\zeta}\right)^{z_{\text{pub-priv}}}},$$

where

$$z_{\text{pub-priv}}:=\eta_1^c\eta_1^y+\eta_2^y, \tag{62}$$

and

$$g_{t+1}=J_{\text{pub-priv}}\left(\frac{1}{g_t}\right)^{\eta_1^y+\eta_2^y}\left\{\zeta\left(x_t^A\right)^{z_{\text{pub-priv}}}+(1-\zeta)\left(\frac{1-\zeta x_t^A}{1-\zeta}\right)^{z_{\text{pub-priv}}}\right\}, \tag{63}$$

where

$$J_{\text{pub-priv}} := \mu \left(\eta_2^c \eta_1^y\right)^{\eta_2^c \eta_1^y} \left(\eta_2^y (1+r)\right)^{\eta_2^y} \left(\frac{\frac{\phi}{1-\eta_2^y}}{1+\gamma \frac{\eta_2^c \eta_1^y}{1+\rho+\gamma\eta_2^y}} \right)^{\eta_2^c \eta_1^y + \eta_2^y} . \tag{64}$$

Having discussed the dynamic systems under different regimes, we start now the analysis of the x^A equilibrium path, since it does not depend upon g for any educational regime. As seen above, the x^A equilibrium path under different educational regimes are quite similar, we could write simply

$$x_{t+2}^A = \Phi(x_t^A), \tag{65}$$

where $\Phi : [0, 1/\zeta] \to [0, \infty)$ is given by

$$\Phi(x) = \frac{1}{\zeta + (1-\zeta)\left(\frac{\frac{1}{x}-\zeta}{1-\zeta}\right)^z} \tag{66}$$

for all $x \in (0, 1/\zeta)$, and extended by continuity to $x = 0$ and $x = 1/\zeta$. Only the parameter $z > 0$ is regime-specific.

It is straightforward to see that, if $z \neq 1$ (and using that $z > 0$), then the fixed points of Φ are 0, 1 and $1/\zeta$, whereas if $z = 1$, then all points in the domain of Φ are fixed. It is also clear that, if the population is homogeneous (i.e., $x_0^A = 1$), then, by $x_1^A = x_0^A$, (65) and (66), x^A will be constant at 1. The following proposition tackles the more interesting case of a heterogeneous population.

Proposition 1. *Given a heterogeneous population (i.e., $x_0^A < 1$), the equilibrium path of x^A converges to 1 if $z < 1$, to 0 if $z > 1$, and is constant if $z = 1$.*

Proof. Let us first note that the equilibrium path of x^A is necessarily constrained to the $(0, 1)$ interval. By definition, $x_0^A \in (0, 1)$ and $x_1^A \in (0, 1)$. The first of these facts implies that $x_t^A \in (0, 1)$ for all even natural t, while the second fact implies that $x_t^A \in (0, 1)$ for all odd natural t. In order to see this, note that, by (65), it is only a matter of checking that, if $x \in (0, 1)$, then also $\Phi(x) \in (0, 1)$.

Let $x \in (0,1)$. That $\Phi(x) > 0$ is clear by (66). Also, if we had $\Phi(x) \geq 1$, then

$$1 \leq \Phi(x) = \frac{1}{\zeta + (1-\zeta)\left(\frac{\frac{1}{x}-\zeta}{1-\zeta}\right)^z}.$$

Since $\zeta \in (0,1)$, this would immediately imply

$$\left(\frac{\frac{1}{x}-\zeta}{1-\zeta}\right)^z \leq 1,$$

and, since $z > 0$, this in turn would yield $x \geq 1$, a contradiction. Therefore, $\Phi(x) \in (0,1)$.

Now, if $z = 1$, then Φ becomes the identity on $[0, 1/\zeta]$ by (66), so that (65) and $x_1^A = x_0^A$ immediately yield a constant x^A.

If $z \neq 1$, then the strategy will be simply to show that x^A is monotonic. This works because, since we have already shown that it is also bounded, it will then converge, and by the continuity of Φ, its limit will be a fixed point of Φ:

$$x^{A*} = \lim_{t\to\infty} x_t^A \Rightarrow x^{A*} = \lim_{t\to\infty} x_{t+2}^A = \lim_{t\to\infty} \Phi(x_t^A) = \Phi(\lim_{t\to\infty} x_t^A) = \Phi(x^{A*}).$$

Since the fixed points of Φ are only 0, 1 and $1/\zeta$, and $1/\zeta > 1$, the limit of x^A will then necessarily be either 0 or 1.

Let us first analyze the $z < 1$ case. We know that $x_1^A = x_0^A \in (0,1)$. Now, if there were a $t \in \{0, 1, ...\}$ such that $x_{t+2}^A \leq x_t^A$, then (65) would yield

$$x_t^A \geq x_{t+2}^A = \frac{1}{\zeta + (1-\zeta)\left(\frac{\frac{1}{x_t^A}-\zeta}{1-\zeta}\right)^z},$$

so that

$$\zeta + (1-\zeta)\left(\frac{\frac{1}{x_t^A}-\zeta}{1-\zeta}\right)^z \geq \frac{1}{x_t^A}.$$

Because $\zeta < 1$, this would imply

$$\left(\frac{\frac{1}{x_t^A}-\zeta}{1-\zeta}\right)^z \geq \frac{\frac{1}{x_t^A}-\zeta}{1-\zeta} > 1,$$

where we have used the fact that $x_t^A \in (0,1)$. Since $z < 1$, this is a contradiction. Therefore, x^A is increasing (in such a way that $x_0^A = x_1^A < x_2^A = x_3^A < x_4^A = ...$), and must converge to 1.

The analysis of the $z > 1$ case is similar. If there were a $t \in \{0, 1, ...\}$ such that $x_{t+2}^A \geq x_t^A$, then, by inverting the directions of the inequalities above, we see that it would be the case that

$$1 < \left(\frac{\frac{1}{x_t^A}-\zeta}{1-\zeta}\right)^z \leq \frac{\frac{1}{x_t^A}-\zeta}{1-\zeta},$$

and since $z > 1$, this is a contradiction. Thus, x^A is decreasing in this case (in such a way that $x_0^A = x_1^A > x_2^A = x_3^A > x_4^A = ...$), and must converge to 0. □

3.1 Inequality

The Gini inequality index at t for this economy with only two wealth (here, human capital) groups can be computed as:

$$G_t = \zeta(1 - x_t^A). \tag{67}$$

Therefore, Proposition 1 implies that, if $z < 1$, then inequality vanishes, if $z > 1$, then inequality converges to ζ, and if $z = 1$, then inequality will be constant at its initial value.

Validity of the condition contained in the following assumption implies that $z < 1$ under all educational regimes, since the largest value of z is $z_{\text{priv-priv}} = \eta_1^c \eta_1^y + \eta_2^c \eta_1^y + \eta_2^y$, as can be gathered from (48), (56), (59) and (62).

Assumption 3. $\eta_1^c \eta_1^y + \eta_2^c \eta_1^y + \eta_2^y < 1$.

Thus, under Assumption 3, Proposition 1 implies that the equilibrium Gini index under all educational regimes converges to 0. However, in the absence of Assumption 3, inequality under some educational regimes would still converge to 0 (which will always be the case under the public-public regime, since $z_{\text{pub-pub}} = \eta_1^c \eta_1^y < 1$), while, under other regimes, it would converge to $\zeta > 0$.

This result is closely related to de la Croix and Doepke (2004). They found that, both under private and public education, there exists a stationary (with respect to auxiliary variables) equilibrium with no inequality. However, under the public education regime, this equilibrium is globally stable, while in the private regime, stability is only local. Thus, considering the latter regime, if the economy at the beginning of time is in a certain sense not close enough to this steady state, then the economy does not converge to such equilibrium, and the income difference might grow. In our model, in the absence of Assumption 3, the completely private regime will present constant or increasing inequality (if $z_{\text{priv-priv}} = 1$ or $z_{\text{priv-priv}} > 1$, respectively), even though, as in de la Croix and Doepke (2004), inequality in the completely public regime always vanishes.

In this and the following subsections, we will compare the four educational regimes regarding (i) economic growth and (ii) speed of reduction of inequality. We start by the latter discussion. Once again, the analysis relies on the value of the parameter z for each educational regime. As we shall see in the following proposition, the lower the value of z, the faster the reduction of inequality.

Proposition 2. *Given a heterogeneous population (i.e., $x_0^A < 1$), under Assumption 3,*

(i) the public-public educational regime is the one with the fastest reduction of inequality;

(ii) the private-private educational regime is the one with the slowest reduction of inequality; and,

(iii) as for the mixed education regimes: (a) if $\eta_1^y \eta_2^c > \eta_2^y$, then inequality in the public-private regime is reduced faster than in the private-public regime; and (b) if $\eta_1^y \eta_2^c < \eta_2^y$, then inequality in the private-public regime is reduced faster than in the public-private regime.

Proof. Since $z_{\text{pub-pub}} < \min\left(z_{\text{priv-pub}}, z_{\text{pub-priv}}\right) \leq \max\left(z_{\text{priv-pub}}, z_{\text{pub-priv}}\right) < z_{\text{priv-priv}}$ and $z_{\text{priv-pub}} \lesseqgtr z_{\text{pub-priv}} \Leftrightarrow \eta_1^y \eta_2^c \lesseqgtr \eta_2^y$, the only thing we need to show here is that the speed of reduction of inequality is inversely related to z.

Given an x^A-equilibrium path, by Assumption 3, even the largest of the values of z, $z_{\text{priv-priv}}$, is lower than 1. Then, from Proposition 1, we know that x^A converges to 1, irrespective of the educational regime in place. We also know, from the proof of Proposition 1, that x^A is increasing. Thus, in order to gauge how fast the convergence of x^A to 1 occurs (and, by (67), how fast inequality is reduced), we may adopt a first-past-the-post criterion, keeping track of the order in which each of the four regimes equilibrium x^A-paths surpasses any fixed mark between x_0^A and 1.

This, in turn, is simply a matter of checking that $\Phi(x)$ decreases with the z-parameter, for any fixed $x \in (0, 1)$:

$$\frac{\partial}{\partial z}\left(\frac{1}{\zeta + (1-\zeta)\left(\frac{\frac{1}{x}-\zeta}{1-\zeta}\right)^z}\right) = -\frac{(1-\zeta)\left(\frac{\frac{1}{x}-\zeta}{1-\zeta}\right)^z \log\left(\frac{\frac{1}{x}-\zeta}{1-\zeta}\right)}{\left(\zeta + (1-\zeta)\left(\frac{\frac{1}{x}-\zeta}{1-\zeta}\right)^z\right)^2} < 0.$$

□

Proposition 2 suggests that the speed of reduction of inequality is highly associated with the elasticities of human capital production. The reason for the completely public regime being the one where inequality vanishes the fastest is that, fixed any educational stage, investment in each student's education is the same. The opposite happens for the private-private regime, where reduction of inequality is the slowest among all regimes (assuming it happens at all, that is, given Assumption 3), since school-time investment at each educational stage is proportional to the relative human capital of parents, as shown in (18) and (19).

With regard to the mixed educational regimes, the speed of reduction of inequality once again depends on adults' basic and advanced education elasticities. The mixed educational regime with the fastest reduction of inequality is the one where it is left to the government to fund the most productive (in terms of the production of human capital) educational stage. If such a stage happens to be the basic one (as suggested by the empirical evidence and to be confirmed by our calibration), then, if only one educational stage is to be publicly funded, publicly funding basic education is more efficient in reducing inequality than publicly funding advanced education.

Note also that, for all educational regimes, the combination of elasticities $\eta_1^c \eta_1^y$ is a common factor in z. As can be seen in (5), this combination of elasticities is associated with parents' human capital. Thus, $\eta_1^c \eta_1^y$ represents the degree of persistence of the human capital of one generation in the human capital production of the next generation.

3.2 Growth

In addition to the analysis of the previous subsection, an investigation of the long-term economic growth rate is also critical: a regime with a more persistent inequality but a higher asymptotic economic growth rate would eventually be doing better than a regime with a faster reduction of inequality but a lower asymptotic economic growth rate, in the sense that the poorest individual of the former economy would eventually become richer than the richest individual of the latter economy.

As in the previous subsection, we here state one more condition on the values of elasticities.

Assumption 4. $\eta_1^y + \eta_2^y < 1$.

Taken in conjunction with Assumption 3, Assumption 4 will ensure that the growth rate of GDP per capita will converge, irrespective of the educational regime in place. In order to reach this conclusion, we will make use of one mathematical claim and two lemmas.

Claim 1. *Given $\alpha \in (0,1)$ and a real sequence $x = (x_n)_{n \in \mathbb{N}}$, if the limit of $x_{n+1} + \alpha x_n$ as $n \to \infty$ exists, then so does the limit of x_n as $n \to \infty$.*

Proof. For all natural n, let us write $y_n := x_{n+1} + \alpha x_n$ and $\lim_{n\to\infty} y_n =: l$.

Given $\varepsilon > 0$, let $\varepsilon' = (1-\alpha)\varepsilon/3$, also positive. By hypothesis, there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$, $|y_n - l| < \varepsilon'$.

Note that we can manipulate the definition of y in order to enable decompositions such as

$$\begin{aligned} x_n &= y_{n-1} - \alpha x_{n-1} \\ &= y_{n-1} - \alpha \left(y_{n-2} - \alpha x_{n-2}\right) \\ &= y_{n-1} - \alpha y_{n-2} + \alpha^2 \left(y_{n-3} - \alpha x_{n-3}\right) \\ &= y_{n-1} - \alpha y_{n-2} + \alpha^2 y_{n-3} - \alpha^3 \left(y_{n-4} - \alpha x_{n-4}\right). \end{aligned}$$

By induction, it is straightforward to see that, for all $n \geq n_1 + 1$,

$$x_n = \sum_{i=0}^{n-n_1-1} (-\alpha)^i y_{n-i-1} + (-\alpha)^{n-n_1} x_{n_1},$$

so that

$$\begin{aligned} x_n - \frac{l}{1+\alpha} &= \sum_{i=0}^{n-n_1-1} (-\alpha)^i y_{n-i-1} + (-\alpha)^{n-n_1} x_{n_1} - \left(\sum_{i=0}^{n-n_1-1} (-\alpha)^i l + \sum_{i=n-n_1}^{\infty} (-\alpha)^i l \right) \\ &= \sum_{i=0}^{n-n_1-1} (-\alpha)^i (y_{n-i-1} - l) + (-\alpha)^{n-n_1} x_{n_1} - \frac{(-\alpha)^{n-n_1} l}{1+\alpha}. \end{aligned}$$

Now, since $\alpha \in (0,1)$, we know that the sequence $(\alpha^n)_{n\in\mathbb{N}}$ converges to 0. Therefore, both $(\alpha^n x_{n_1}/\alpha^{n_1})_{n\in\mathbb{N}}$ and $(\alpha^n l/((1+\alpha)\,\alpha^{n_1}))_{n\in\mathbb{N}}$ also converge to 0, so that there exists a natural n_0 (without loss of generality strictly greater than n_1), such that, for all $n \geq n_0$, $|\alpha^{n-n_1} x_{n_1}| < \varepsilon/3$ and $|\alpha^{n-n_1} l/(1+\alpha)| < \varepsilon/3$.

Thus, for any $n \geq n_0$, the triangle inequality gives

$$\begin{aligned} \left| x_n - \frac{l}{1+\alpha} \right| &\leq \sum_{i=0}^{n-n_1-1} \alpha^i |y_{n-i-1} - l| + \left|\alpha^{n-n_1} x_{n_1}\right| + \left| \frac{\alpha^{n-n_1} l}{1+\alpha} \right| \\ &< \sum_{i=0}^{n-n_1-1} \alpha^i \varepsilon' + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \sum_{i=0}^{\infty} \alpha^i \varepsilon' + \frac{2\varepsilon}{3} \\ &= \frac{\varepsilon'}{1-\alpha} + \frac{2\varepsilon}{3} = \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

Lemma 1. *Under Assumptions 3 and 4, under all four educational regimes, the growth factor of average human capital g converges.*

Proof. The dynamics for g in (55), (57), (60) and (63) can all be expressed as:

$$g_{t+1} g_t^{\eta} = J \left\{ \zeta \left(x_t^A\right)^z + (1-\zeta) \left(\frac{1-\zeta x_t^A}{1-\zeta} \right)^z \right\}$$

for all $t \in \{0, 1, ...\}$, where not only J and z are regime-specific, but also η: it equals η_1^y in the public-public and in the private-public regimes, and $\eta_1^y + \eta_2^y$ in the remaining regimes. Thus, by Assumption 4, in any of the educational regimes, $\eta \in (0, 1)$.

From the remark made right before Proposition 1, we know that, if the population is homogeneous (i.e., $x_0^A = 1$), then $x^A \equiv 1$. If the population is heterogeneous, then, by Assumption 3 and Proposition 1, x^A will also converge to 1. Thus, in either case, since $z > 0$, the above convex combination within curly brackets converges to 1.

Therefore, $\lim_{t\to\infty} g_{t+1} g_t^\eta = J$, so that, by the continuity of $\log$, $\lim_{t\to\infty}(\log g_{t+1} + \eta \log g_t) = \log J$. By Claim 1, $\log \circ g$ converges, whence, by the continuity of $\exp$, g converges too. $\square$

Under Assumptions 3 and 4, we already know that g converges under any educational regime. By (55), (57), (60) and (63), the long-term growth factor of the average human capital, for each educational regime, is:

$$g^*_{\text{priv-priv}} = (J_{\text{priv-priv}})^{\frac{1}{1+\eta_1^y+\eta_2^y}}, \tag{68}$$

$$g^*_{\text{pub-pub}} = (J_{\text{pub-pub}})^{\frac{1}{1+\eta_1^y}}, \tag{69}$$

$$g^*_{\text{priv-pub}} = (J_{\text{priv-pub}})^{\frac{1}{1+\eta_1^y}} \tag{70}$$

and

$$g^*_{\text{pub-priv}} = (J_{\text{pub-priv}})^{\frac{1}{1+\eta_1^y+\eta_2^y}}. \tag{71}$$

Lemma 2. *Under Assumptions 3 and 4, under all four educational regimes, the growth factor of GDP per capita converges to g^* as given in the above expressions.*

Proof. Since κ is constant in equilibrium due to the constancy of the international interest rate, the growth factor of GDP is $Y_{t+1}/Y_t = L_{t+1}/L_t$. Using (10), since the fertility choice under all educational regimes is constant over time and group-invariant, we can write L_{t+1}/L_t as

$$\frac{L_{t+1}}{L_t} = \frac{(1-\phi n)\sum_{i\in\{A,B\}} P_{t+1}^{a,i} h_{t+1}^{a,i} - n\bar{h}_{t+1}\left(\sum_{i\in\{A,B\}} P_{t+1}^{a,i} e_{t+1}^{c,i} + \sum_{i\in\{A,B\}} P_{t}^{a,i} e_{t+1}^{y,i}\right)}{(1-\phi n)\sum_{i\in\{A,B\}} P_{t}^{a,i} h_{t}^{a,i} - n\bar{h}_{t}\left(\sum_{i\in\{A,B\}} P_{t}^{a,i} e_{t}^{c,i} + \sum_{i\in\{A,B\}} P_{t-1}^{a,i} e_{t}^{y,i}\right)}. \tag{72}$$

As seen above, L_{t+1}/L_t depends on adults' decision rules for fertility and school-time investment (or government's decision rules for school-time investment, in the case of a publicly funded educational stage). That is, the growth factor of GDP, L_{t+1}/L_t, is dependent upon the educational regime in place.

Private-private regime:

Plugging the school-time investment decisions (18) and (19) in (72), we find

$$\frac{L_{t+1}}{L_t} = \frac{(1-\phi n)\sum_{i\in\{A,B\}} P_{t+1}^{a,i} h_{t+1}^{a,i} - \frac{\eta_2^c\eta_1^y\phi n\bar{h}_{t+1}}{1-\eta_2^y-\eta_2^c\eta_1^y}\sum_{i\in\{A,B\}} P_{t+1}^{a,i} x_{t+1}^{i} - \frac{\eta_2^y(1+r)\phi n\bar{h}_{t}}{1-\eta_2^y-\eta_2^c\eta_1^y}\sum_{i\in\{A,B\}} P_{t}^{a,i} x_{t}^{i}}{(1-\phi n)\sum_{i\in\{A,B\}} P_{t}^{a,i} h_{t}^{a,i} - \frac{\eta_2^c\eta_1^y\phi n\bar{h}_{t}}{1-\eta_2^y-\eta_2^c\eta_1^y}\sum_{i\in\{A,B\}} P_{t}^{a,i} x_{t}^{i} - \frac{\eta_2^y(1+r)\phi n\bar{h}_{t-1}}{1-\eta_2^y-\eta_2^c\eta_1^y}\sum_{i\in\{A,B\}} P_{t-1}^{a,i} x_{t-1}^{i}},$$

where n is given in (16).

Using (1), (9), (14), (15), the above identity can be simplified to

$$\frac{L_{t+1}}{L_t} = \sqrt{n}g_{t-1}\frac{g_t\left(\left(1-\eta_2^y-\eta_2^c\eta_1^y\right)-\left(1-\eta_2^y\right)\phi n\right)-\eta_2^y\left(1+r\right)\phi\sqrt{n}}{g_{t-1}\left(\left(1-\eta_2^y-\eta_2^c\eta_1^y\right)-\left(1-\eta_2^y\right)\phi n\right)-\eta_2^y\left(1+r\right)\phi\sqrt{n}}. \tag{73}$$

Public-public regime:

For the completely public regime, (72) allows us to write the growth factor of GDP as

$$\frac{L_{t+1}}{L_t} = \frac{(1-\phi n)\sum_{i\in\{A,B\}} P_{t+1}^{a,i} h_{t+1}^{a,i} - n\bar{e}^c \bar{h}_{t+1} \sum_{i\in\{A,B\}} P_{t+1}^{a,i} - n\bar{e}^y \bar{h}_{t+1} \sum_{i\in\{A,B\}} P_t^{a,i}}{(1-\phi n)\sum_{i\in\{A,B\}} P_t^{a,i} h_t^{a,i} - n\bar{e}^c \bar{h}_t \sum_{i\in\{A,B\}} P_t^{a,i} - n\bar{e}^y \bar{h}_t \sum_{i\in\{A,B\}} P_{t-1}^{a,i}}, \tag{74}$$

where n, $\bar{e}^c$, $\bar{e}^y$ are constant over time, group-invariant, and given by (25), (27), (28), respectively.

Using (1), (9) and (15), (74) can be rewritten as

$$\frac{L_{t+1}}{L_t} = \sqrt{n} g_t. \tag{75}$$

Private-public regime:

Following the same procedure, we can write L_{t+1}/L_t under the private-public regime as

$$\frac{L_{t+1}}{L_t} = \sqrt{n} g_t, \tag{76}$$

where n is constant over time, group-invariant, and given by (33).

Public-private regime:

Following the same procedure, we can rewrite L_{t+1}/L_t under the private-public regime as

$$\frac{L_{t+1}}{L_t} = \sqrt{n} g_{t-1} \frac{g_t((1-\phi n) - n\bar{e}^c) - \sqrt{n} e^y}{g_{t-1}\left((1-\phi n) - n\bar{e}^c\right) - \sqrt{n} e^y}, \tag{77}$$

where n and $\bar{e}^c$ are constant over time, group-invariant, and given by (41), (44), respectively, and e^y is the constant part of (46):

$$e^y := \frac{\phi \eta_2^y (1+r)}{1-\eta_2^y} \frac{1+\rho+\gamma\eta_2^y}{1+\rho+\gamma\eta_2^y+\gamma\eta_2^c\eta_1^y}. \tag{78}$$

Let P_t be the total population of the economy at period t, which includes nP_t^a children, nP_{t-1}^a young individuals, P_t^a adults and P_{t-1}^a senior citizens:

$$P_t := (1+n)\left(P_t^a + P_{t-1}^a\right) \tag{79}$$

Using (79) and (1), we can evaluate the growth factor of GDP per capita as

$$\frac{\frac{Y_{t+1}}{P_{t+1}}}{\frac{Y_t}{P_t}} = \frac{Y_{t+1}}{Y_t}\frac{P_t}{P_{t+1}} = \frac{L_{t+1}}{L_t}\frac{P_t}{P_{t+1}} = \frac{L_{t+1}}{L_t}\frac{1}{\sqrt{n}} \tag{80}$$

Under Assumptions 3 and 4, Lemma 1 states that g converges to g^*. Hence, from (73), (75), (76), (77), we see that L_{t+1}/L_t also converges to $\sqrt{n}g^*$. Therefore,

$$\lim_{t\to\infty}\left(\frac{\frac{Y_{t+1}}{P_{t+1}}}{\frac{Y_t}{P_t}}\right) = \frac{1}{\sqrt{n}}\lim_{t\to\infty}\left(\frac{L_{t+1}}{L_t}\right) = \frac{1}{\sqrt{n}}\sqrt{n}g^* = g^*.$$

□

Evidently, once we are assured of the asymptotic equivalence between the growth factor of average human capital and the growth factor of GDP per capita, the same applies with respect to their growth rates.[5]

Proposition 3. *Under Assumptions 3 and 4, under all four educational regimes, the long-term growth rate of GDP per capita is invariant to the elasticities of the human capital produced during childhood with respect to parental human capital (η_1^c) and teacher quality (η_3^c).*

Proof. This is just a matter of looking at (68), (69), (70), (71), (49), (58), (61) and (64) and applying Lemma 2. □

The next proposition shows that, in the long run, the economy must grow faster if basic education is private, rather than public. Since basic education sets the stage for productive advanced education, in what regards economic growth, it is more advantageous to let parents make private decisions about investing in basic schooling.

Proposition 4. *Under Assumptions 3 and 4, fixed any type of advanced education, the long-term growth rate of GDP per capita is higher if basic education is private:*

(i) $g^*_{\text{priv-priv}} > g^*_{\text{pub-priv}}$*; and*

(ii) $g^*_{\text{priv-pub}} > g^*_{\text{pub-pub}}$.

[5] Growth rate = growth factor − 1.

Proof. Since Assumptions 3 and 4 hold, we know from Lemma 1 that g converges under all four regimes, the limits being given in (68), (69), (70) and (71). These coincide with the long-term growth factors of GDP per capita, by Lemma 2. From these expressions, it is clear that, in order to proceed with the ranking of growth rates (or, equivalently, growth factors), it suffices to show that $J_{\text{priv-priv}} > J_{\text{pub-priv}}$ and $J_{\text{priv-pub}} > J_{\text{pub-pub}}$.

If $J_{\text{priv-priv}} \leq J_{\text{pub-priv}}$, we would have, by (49) and (64),

$$\frac{\phi}{1-\eta_2^c\eta_1^y-\eta_2^y} \leq \frac{\frac{\phi}{1-\eta_2^y}}{1+\gamma\frac{\eta_2^c\eta_1^y}{1+\rho+\gamma\eta_2^y}},$$

so that

$$1+\gamma\frac{\eta_2^c\eta_1^y}{1+\rho+\gamma\eta_2^y} \leq \frac{1-\eta_2^c\eta_1^y-\eta_2^y}{1-\eta_2^y} = 1-\frac{\eta_2^c\eta_1^y}{1-\eta_2^y},$$

a contradiction since the left-hand side of this inequality is greater than 1, while the right-hand side is lower than 1.

As for the second inequality, let $\mathcal{J} : [0, \eta_2^c\eta_1^y] \to (0, \infty)$ be given by

$$\mathcal{J}(q) = \left(1+(1-\eta_1^y)\sqrt{\frac{\gamma(1-q)}{\phi(1+\rho+\gamma)}}\right)^{\eta_2^y}\left(\frac{\frac{\phi(1+\rho+\gamma q)}{\left(1+\rho+\gamma\eta_2^c\eta_1^y\right)(1-q)}}{1+\gamma\eta_2^y\frac{1-\eta_1^y+\sqrt{\frac{\phi(1+\rho+\gamma)}{\gamma(1-q)}}}{1+\rho+\gamma\eta_2^c\eta_1^y}}\right)^{\eta_2^c\eta_1^y+\eta_2^y},$$

so that $\mathcal{J}(0) = J_{\text{pub-pub}}/\left[\mu\left(\eta_2^c\eta_1^y\right)^{\eta_2^c\eta_1^y}\left(\eta_2^y\right)^{\eta_2^y}\right]$ and $\mathcal{J}(\eta_2^c\eta_1^y) = J_{\text{priv-pub}}/\left[\mu\left(\eta_2^c\eta_1^y\right)^{\eta_2^c\eta_1^y}\left(\eta_2^y\right)^{\eta_2^y}\right]$. If $\mathcal{J}$ is strictly increasing (or, equivalently, $\log\circ\mathcal{J}$), then we are done. And, in fact, for any $q \in (0, \eta_2^c\eta_1^y)$, we have

$$\begin{aligned}\frac{d}{dq}\left(\log\left(\mathcal{J}(q)\right)\right) &= \eta_2^y\frac{(1-\eta_1^y)\sqrt{\frac{\gamma}{\phi(1+\rho+\gamma)}}\left(-\frac{1}{2\sqrt{1-q}}\right)}{1+(1-\eta_1^y)\sqrt{\frac{\gamma(1-q)}{\phi(1+\rho+\gamma)}}} \\ &\quad+\left(\eta_2^c\eta_1^y+\eta_2^y\right)\left(\frac{\gamma}{1+\rho+\gamma q}+\frac{1}{1-q}-\frac{\gamma\eta_2^y\frac{\sqrt{\frac{\phi(1+\rho+\gamma)}{\gamma}}}{1+\rho+\gamma\eta_2^c\eta_1^y}\frac{1}{2(1-q)\sqrt{1-q}}}{1+\gamma\eta_2^y\frac{1-\eta_1^y+\sqrt{\frac{\phi(1+\rho+\gamma)}{\gamma(1-q)}}}{1+\rho+\gamma\eta_2^c\eta_1^y}}\right).\end{aligned}$$

After several algebraic manipulations, this can be seen to coincide with

$$
\begin{aligned}
&\left(\eta_2^c \eta_1^y + \eta_2^y\right)\left(\frac{\gamma}{1+\rho+\gamma q} + \frac{1}{2(1-q)}\right) \\
&+\frac{1}{2(1-q)} \frac{\eta_2^c \eta_1^y \left(1-\eta_1^y\right) + \frac{\left(\eta_2^c\eta_1^y+\eta_2^y\right)\left(1+\rho+\gamma\eta_2^c\eta_1^y\right)+\gamma\left(1-\eta_1^y\right)\eta_2^c\eta_1^y\eta_2^y}{1+\rho+\gamma\left(\eta_2^c\eta_1^y+\left(1-\eta_1^y\right)\eta_2^y\right)}\sqrt{\frac{\phi(1+\rho+\gamma)}{\gamma(1-q)}}}{\left(1-\eta_1^y+\sqrt{\frac{\phi(1+\rho+\gamma)}{\gamma(1-q)}}\right)\left(1+\frac{\gamma\eta_2^y}{1+\rho+\gamma\left(\eta_2^c\eta_1^y+\left(1-\eta_1^y\right)\eta_2^y\right)}\sqrt{\frac{\phi(1+\rho+\gamma)}{\gamma(1-q)}}\right)},
\end{aligned}
$$

the positivity of which can be noted at a glance (the minus sign only enters this expression through terms of the form $1-q$ and $1-\eta_1^y$, both positive).

□

Economic growth rates under all educational regimes can be compared two by two. By Proposition 4, there are but six possible rankings of educational regimes by their economic growth:

- ranking #1: $g^*_{\text{priv-priv}} > g^*_{\text{pub-priv}} \geq g^*_{\text{priv-pub}} > g^*_{\text{pub-pub}}$;
- ranking #2: $g^*_{\text{priv-pub}} \geq g^*_{\text{priv-priv}} > g^*_{\text{pub-priv}} \geq g^*_{\text{pub-pub}}$;
- ranking #3: $g^*_{\text{priv-pub}} > g^*_{\text{pub-pub}} \geq g^*_{\text{priv-priv}} > g^*_{\text{pub-priv}}$;
- ranking #4: $g^*_{\text{priv-priv}} \geq g^*_{\text{priv-pub}} > g^*_{\text{pub-priv}} \geq g^*_{\text{pub-pub}}$;
- ranking #5: $g^*_{\text{priv-priv}} \geq g^*_{\text{priv-pub}} > g^*_{\text{pub-pub}} \geq g^*_{\text{pub-priv}}$; and
- ranking #6: $g^*_{\text{priv-pub}} \geq g^*_{\text{priv-priv}} > g^*_{\text{pub-pub}} \geq g^*_{\text{pub-priv}}$.

In order to determine which of these rankings prevails, we must have a better idea of the values of the parameters of our model, an issue addressed in the next section.

4 Calibration

To gain some insight into the comparison between economic growth rates under different educational regimes, we calibrate the model's parameter values using data from the Organisation for

Economic Co-operation and Development (OECD) economies. We have chosen to work with this set of countries due to the large availability of educational information. Annually the OECD releases the Education at a Glance report, from which it is possible to trace the educational profile of OECD member countries.

We calibrate the model under the assumption that one period has a length of 15 years.[6] Our sample is composed of thirty OECD member countries. In order to classify these countries according to adopted educational regime, we have considered enrollment rates by type of institution in basic (elementary and secondary) and advanced education (higher education). On average, during the period considered, 87.5% of enrollments at the basic stage were in public institutions, with a standard deviation of 11.5%. Ireland had the highest percentage of pupils studying at public institutions (99.3%), while Belgium has the lowest rate (44.4%). At the advanced education stage, 77% of enrollments were in public institutions, with a standard deviation of 20.5%. Greece had the highest percentage of higher education students in public universities (99.3%), while South Korea had the lowest rate (19.6%).

It is obviously challenging to propose a clear-cut classification for these countries, since, for all of them, at each educational stage, both types of school systems coexist. For each of our two educational stages, we have decided to call the school system in that stage public if the average enrollment rate in public institutions of that stage over the last fifteen years available in the data (2003 - 2017) exceeded 50%.[7] In this way, twenty-three countries were classified as adopting the public-public regime, three as adopting the public-private regime, and one as adopting the private-private regime.[8]

To estimate the time spent by parents to raise a child, we make use of the empirical evidence suggested by Fedick et al. (2005) and Folbre and Yoon (2007), based on the Time Use Surveys.

[6] We consider only countries that were OECD members during the whole analysis period.

[7] Even if a more rigorous threshold, such as 65%, were chosen, this would cause no change to the classification of countries.

[8] Countries classified as public-public: Australia, Austria, Czech Republic, Denmark, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Italy, Mexico, New Zealand, Norway, Poland, Portugal, Slovak Republic, Spain, Sweden, Switzerland, Turkey, United States. Countries classified as public-private: Japan, Korea, United Kingdom. Country classified as private-private regime: Belgium. We could not classify three countries because they present missing information (for several years) during this period: Canada, Netherlands, Luxembourg.

According to these authors, Canadian and American employed parents spend an average of 7.4 hours per parent per day taking care of and looking after their children. This leads to $\phi = 0.31$. As for the discount factor ρ, although it does affect choices, it will not influence the ranking of growth rates. We set it equal to 0.67 (equivalent to 0.993 per quarter).

The parameters γ, η_2^c, η_1^y, η_2^y and μ are chosen as to match some statistics (fertility, instruction time and GDP per capita) for OECD countries. We determine all the parameter value estimates using data from countries classified as having a public-public educational regime and those with a public-private regime. For all of these countries, we computed the average fertility rate per person over a fifteen-year cycle. According to World Bank data, from 2003 to 2017, the countries classified as public-public showed an average fertility rate of 0.85 per person (or 1.69 per woman), while those classified as public-private experienced a slightly lower fertility rate per person, 0.73 (or 1.46 per woman), in line with our model.

We also compute the average time invested in public basic education per child for each group of countries. To this purpose, we use the data of compulsory instruction time in public institutions at the basic educational stage, available at the Education at a Glance report. This data underwent a methodological change in 2014. Only from 2014 onward, instruction time began to be disclosed by educational stage. However, for the years 2015, 2016 and 2017, there are missing data for two of the three public-private regime countries in our dataset.[9] Therefore, we chose to use only data from 2014 to estimate time investment in basic education. As these data refer to the educational regime, we consider that this information is relatively constant over time and does not experience abrupt changes. For those countries classified as having a completely public educational regime, this procedure resulted in an average total of 8,353 hours per child. For the public-private educational regime countries, we found the total of 8,623 hours per child in basic education.

Our goal is to calibrate the parameters γ, η_2^c, η_1^y and η_2^y in such a way that (25), (41), (27) and (44) are close to the empirical evidence of fertility rates per person and instruction time per child during basic education. The remaining parameter, the aggregate multiplicative constant μ, is

[9] Data from Korea and the United Kingdom are missing.

calibrated from the long-term human capital growth rate of public-public regime countries, (69). To this end, we use GDP per capita growth rates for countries classified as public-public, from 2003 to 2017. This subset of countries showed an average growth rate of GDP per capita of 21.7% during this period.[10]

Table 1: Calibrated parameter values

Parameter	Description	Value
γ	Altruism factor	0.70
η_2^c	Elasticity of human capital produced during childhood with respect to investment in basic education	0.35
η_1^y	Elasticity of human capital produced during youth with respect to human capital produced during childhood	0.69
η_2^y	Adults' advanced education elasticity	0.11
μ	Aggregate multiplicative constant	3.88

It may be noted that these parameter values easily satisfy all of the aforementioned assumptions. They are also in line with those in the literature. For instance, the fact that adults' basic education elasticity $\eta_1^y \eta_2^c \approx 0.24$ is more than twice as high as adults' advanced education elasticity $\eta_2^y = 0.11$ parallels a finding of Cunha and Heckman (2010) that the productivity of investment in education is decreasing also throughout childhood educational stages.[11]

In Table 2, we show how our model fits the empirical data. To calculate the growth rate path under the public-private regime, we use the interest rate accumulated from 2003 to 2017. On average the accumulated interest rate of OECD member countries was 83.7%, which is equivalent to 4.1% per year. For all empirical data considered in this calibration, our model generates values within one standard deviation of the sample mean. Comparing our estimates of time investment in education to those found in the calibration of de la Croix and Doepke (2004), our model generates values closer to the empirical data.[12]

As commented earlier, it follows from Proposition 4 that there are only six possible rankings of educational regimes by economic growth. We can now use our calibrated parameter values to

[10]The calibrated value of μ is robust to the choice of which regimes to use in the calibration process. Had we chosen to use the growth rate of GDP per capita of public-private countries, the calibrated μ would be 3.82.

[11]They suggest that, while the productivity of investments in education at the first stage of childhood is around 0.23, at the second stage it is only 0.02.

[12]The calibration of de la Croix and Doepke (2004) generates the instruction time per child of only 2.5% of fifteen years, that is, approximately 3,300 hours.

Table 2: Model fit

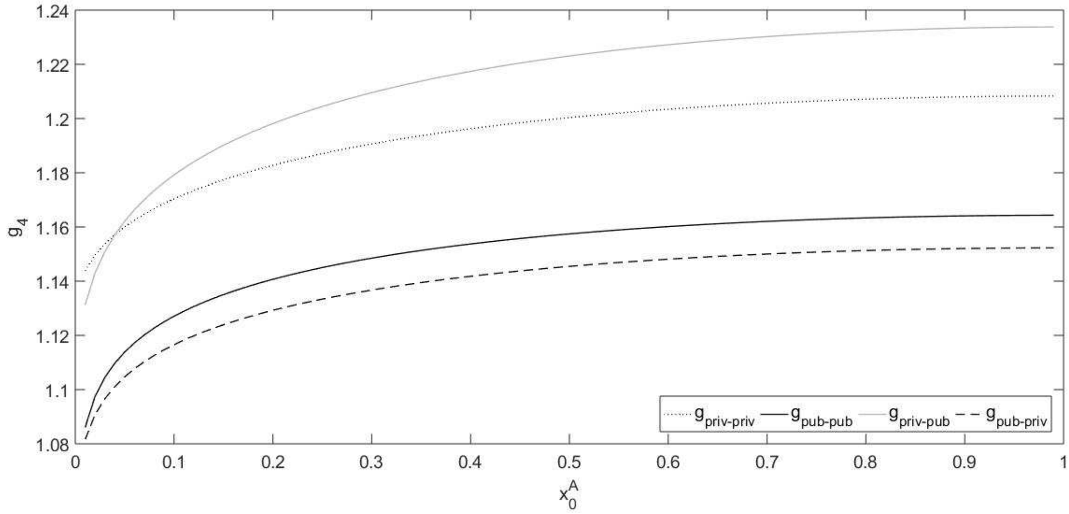
	Public-public regime			**Public-private regime**		
	Average	SD	Model	Average	SD	Model
Fertility rate per person	0.84	0.16	0.95	0.73	0.17	0.84
Instruction time per child	6.3%	1.8%	4.5%	6.7%	1.0%	7.7%
Growth rate of GDP per capita	21.7%	20.9%	21.7%	25.9%	23.1%	27.1%

check which of these rankings should be expected to occur in practice. In Figure 3, the value of g_t under each educational regime is plotted, for different initial relative human capital levels x_0^A, and for two different periods: $t = 4, 10$ (i.e., 60 and 150 years). As we can see, only in the short run does the initial relative human capital affect the ranking of economic growth rates. For $t = 10$, this ranking is invariant to the initial relative human capital.

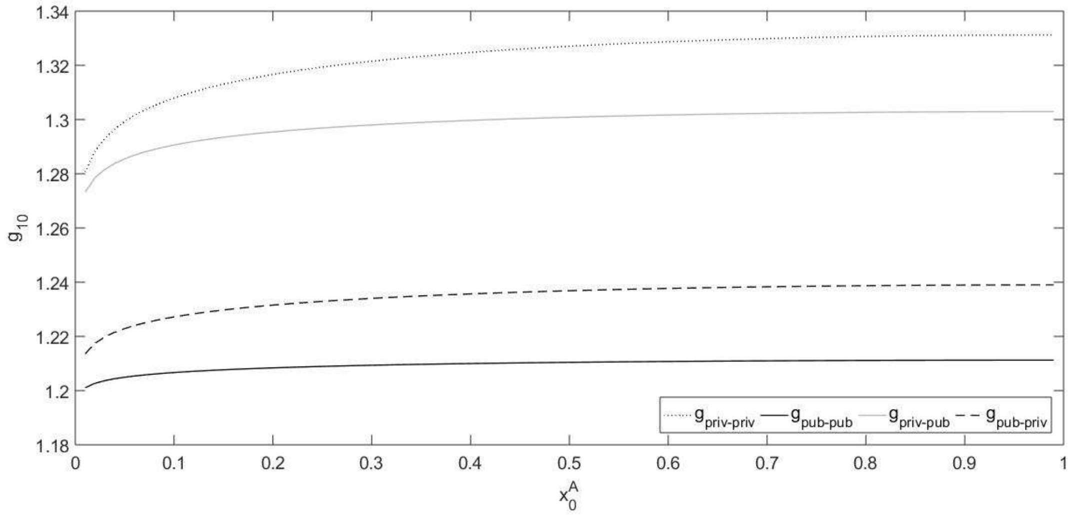
In Figure 4, we plot g for each educational regime over time, considering $x_0^A = 0.5$. Making use of the calibrated parameter values, in the long run the ordering of economic growth rates under different educational regimes follows ranking #4. Therefore, the economy under the private-private educational regime grows faster than any other. Although the completely public regime is the one under which inequality vanishes the fastest, regarding long-term economic growth, it is the one with the lowest economic growth rate. Therefore, comparing only completely educational regimes, our model generates a tradeoff between economic growth and reduction of inequality.

This tradeoff also holds when it is the mixed regimes that are stacked up against each other. According to ranking #4, an economy grows faster under the private-public educational regime than under the public-private one. However, regarding the reduction of inequality, as already mentioned, our calibration suggests that investment in basic education has the highest productivity for human capital formation, $\eta_1^y \eta_2^c > \eta_2^y$. Therefore, by Proposition 2, public funding of basic education is more efficient in reducing inequality than public funding of advanced education, that is, inequality vanishes faster under the public-private regime than under the private-public one.

It is worth noting that the comparison among mixed regimes depends not only on elasticities of human capital, but also on the interest rate. In Figure 5, we plot g for each educational regime



(a) $t = 4$



(b) $t = 10$

Figure 3: Initial relative human capital and growth factor of average human capital

over time considering an annual interest rate of 7%. In this high interest rate scenario, the tradeoff among mixed educational regimes no longer holds. The ordering of long-term economic growth rates under different educational regimes follows ranking #1. Economies under the public-private regime present higher growth and faster reduction of inequality. As adults need to save in order to afford old-age spending on consumption and send their kids to college at the same time, as

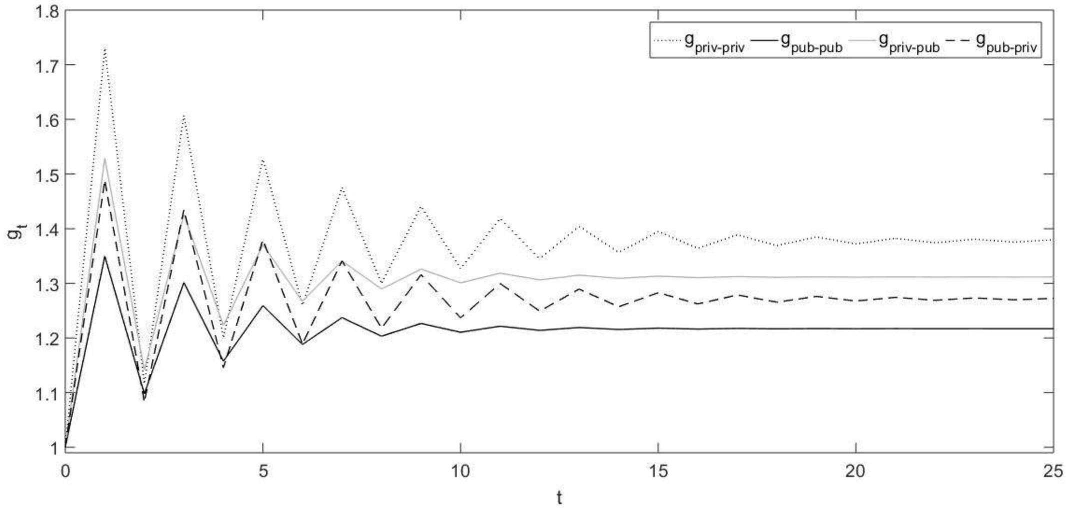


Figure 4: Growth factor of average human capital over time

the interest rate increases, so does income from savings. In this case, if adults invest privately in advanced, rather than basic, education, production of human capital will be higher, even if the advanced educational stage is less productive than the basic one. Thus, if society were to choose only one educational stage to fund publicly, it should probably be the basic one, both out of growth and inequality reduction concerns.

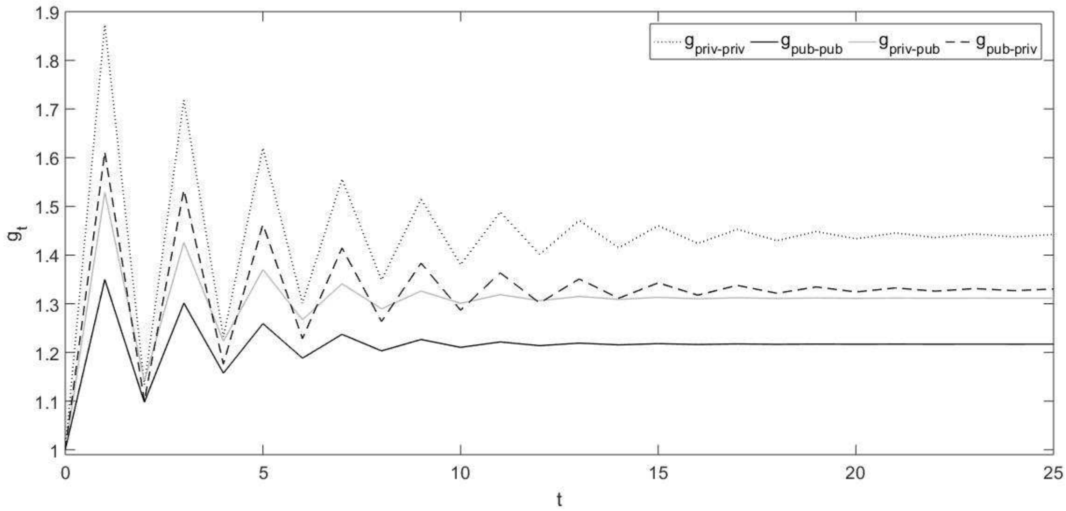


Figure 5: Growth factor of average human capital over time (interest rate of 7% per year)

Lastly, comparing short-term growth factors of average human capital, we note that other patterns could emerge. As can be seen in Figures 3 and 4, for a short period of time, it would even be possible that the economy under the completely private regime grows slower than the economy under other educational regimes.

5 Conclusion

We have explored the impacts of different educational regimes on economic growth and the dynamics of inequality in a framework that accounts for the hierarchical nature of education. The analysis leads to two main conclusions. First, inequality is reduced faster in the completely public regime than in any other. The opposite holds when both educational stages are private, so that inequality is reduced at the slowest rate of all regimes. With respect to the mixed educational regimes, publicly funding basic education (considering that adults' basic education elasticity is greater than adults' advanced education elasticity) leads to a faster reduction of inequality.

Also, considering mixed educational regimes, in order to obtain a faster reduction of inequality, governments should fund .

Second, regarding economic growth, the regime with the highest long-term growth rate necessarily has private basic education. In other words, if the goal of a society is only to maximize economic growth, then this educational stage should be private. Thus, although, under a completely public regime, inequality vanishes the fastest, this is never the regime with the highest growth in the long run.

In the long run, the regimes with highest growth rates necessarily have private basic education. Also, according to both orderings #4 and #1, the completely private regime presents the highest economic growth and the completely public regime presents the lowest one. Therefore, comparing only the completely public and the completely private regimes, there is a tradeoff between economic growth and the reduction of inequality.

Comparisons between the mixed educational regimes are dependent on the elasticities of human capital and on the interest rate. For the average interest rate of OECD countries, we have seen that a tradeoff between economic growth and inequality reduction is present. On one hand, if basic education is the most productive one, as suggested in the empirical literature and by our calibration, then it should be public, in order to generate a faster reduction of inequality. On the other hand, in order to reach higher economic growth, basic education should be private. Nonetheless, if the interest rate is high, this tradeoff disappears, and it is basic education that should be funded publicly in order to bring a faster reduction of inequality and higher economic growth.

This study lends itself to various possibilities of extensions, for which the results above would not be secured. For instance, young individuals could be allowed to work and use their labor income to finance advanced education. In our model, parents need to save in order to finance the education of the young. Thus, the savings interest rate, which is exogenously determined, affects only the growth rate of economies where advanced education is private and, therefore, influences the growth-rate differentials between regimes. Besides that, one could leave also to parents the decision of how much time to spend raising their children.

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